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# SCALING AND UNIVERSALITY IN STATISTICAL PHYSICS

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The twin concepts of Scaling and Universality have played an important role in the description of statistical systems. Hydrodynamics contains many applications of scaling including descriptions of the behavior of boundary layers (Prandtl, Blasius) and of the fluctuating velocity in turbulent flow (Kolmogorov, Heisenberg, Onsager).

Phenomenological theories of behavior near critical points of phase transitions made extensive use of both scaling, to define the size of various fluctuations, and universality to say that changes in the model would not change the answers. These two ideas were combined via the statement that elimination of degrees of freedom and a concomitant scale transformation left the answers quite unchanged. In Wilson's hands, this mode of thinking led to the renormalization group approach to critical phenomena.

Subsequently, Feigenbaum showed how scaling, universality, and renormalization group ideas could be applied to dynamical systems. Specifically, this approach enabled us to see how chaos first arises in those systems in which but a few degrees of freedom are excited. In parallel Libchaber developed experiments aimed at understanding the onset of chaos, the results of which were subsequently used to show that Feigenbaum's universal behavior was in fact realized in honest-to-goodness hydrodynamical systems. More recently, Gemunu Gunaratne, Mogens Jensen, and Itamar Procaccia have indicated that they believe that a different (and weaker) universality might hold for the fully chaotic behavior of low dimensional dynamical systems.

Dynamically generated situations often seem to show kinds of scaling and universality quite different from that seen in critical phenomena. A technical difference which seems to arise in these intrinsically dynamical processes is that instead of having a denumerable list of different critical quantities, each with their critical index, instead there is continuum of critical indices. This so-called multifractal behavior may nonetheless show some kinds of universality. And indeed this might be the kind of scaling and universality shown by those hydrodynamical systems in which many degrees of freedom are excited.

## 1. Scaling in hydrodynamics: early developments

Concepts of scaling, which are an extension of Fourier's [1] idea of dimensional analysis, reached a very high degree of refinement in the study of hydrodynamics. So I start my story from one of the simplest set of equations which are used to describe fluid flow [2], the Navier–Stokes equations

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -(\nabla p)/\rho + \nu\nabla^2\boldsymbol{u}\,, \\ \nabla\cdot\boldsymbol{u} &= 0\,. \end{aligned} \tag{1.1}$$

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These equations assume an incompressible fluid in which $\boldsymbol{u}$ represents the local fluid velocity, $p$ and $\rho$ are respectively its pressure and density, and $\nu$ is its kinematic viscosity.

There are two kinds of scaling solutions to eqs. (1.1). One kind has as its prototype the Blasius description [3] of a two dimensional time-independent flow past a semi-infinite flat plate. The velocity at infinity is parallel to the plate and has a magnitude $U$. The coordinate along the plate, $x$, is used to set the scale for the problem. Then suitable simplifying approximations are made in which small terms are eliminated from the Navier–Stokes equation. Specifically, one says that the variation in $x$ is much slower than that in $y$ and that the pressure term may be neglected in the $x$-component of the Navier–Stokes equation. Finally, a scaling or similarity solution is worked out in which each variable is expressed as an appropriately chosen dimensional constant times a suitable function of the dimensionless variables left in the problem. In this example, we write

$$\begin{aligned} u_x &= Ug^*(y/Y(x)) , \\ u_y &= U(Y(x)/x)h^*(y/Y(x)) , \\ Y(x) &= (\nu x/U)^{1/2} . \end{aligned} \tag{1.2}$$

Here $Y(x)$ sets the scale for $y$ while $g^*$ and $h^*$ are unknown functions which are determined by eqs. (1.1).

Kolmogorov [4], Heisenberg [5] and Onsager [6] analyzed turbulent flow by applying a more sophisticated kind of scaling to eq. (1.1). Turbulence involves a complicated flow pattern in which there are velocity fluctuations over a wide range of length scales. Let $b$ denote a particular length scale and $u_b$ describe the typical variation in velocity in that scale. In a turbulence problem, the kinematic viscosity is often a very small quantity. To express this smallness, form the dimensionless number

$$\mathrm{Re} = UL/\nu , \tag{1.3}$$

called the Reynolds number from a typical length scale of the enclosure, $L$, and a typical velocity of large-scale motion, $U$. It is quite often true that Re is very large indeed. (For an airplane flying at jet speeds Re is $10^9$ or so.) Nonetheless the viscosity cannot be ignored because it provides the sole mechanism by which the kinetic energy which enters the fluid may be dissipated. Turbulence is then described by using a scaling argument, now in a statistical form, that says that kinetic energy is cascaded downward from longer wavelengths to shorter ones. Notice that a typical contribution to the kinetic energy density from terms at scale $b$ is $u_b^2$ while the current or flux of kinetic energy at this scale is $u_b^3$. Of course, the divergence of a current (here of order $(u_b^3/b)$ is a rate of change of the corresponding density produced by the downward cascade. Since the process is steady, the same amount is carried downward at all scales and hence

$$(u_b^3)/b = \varepsilon \tag{1.4}$$

with $\varepsilon$ being a constant independent of the scale, $b$. Eq. (1.4) can then be converted into an estimate of $u_b$ namely

$$u_b = (b/L)^{1/3} U \ . \tag{1.5}$$

This statement has proven to give a roughly correct indication of the typical velocities encountered in turbulent flow. It works in the socalled inertial range, i.e. for scales below $L$ and above the Kolmogorov length, $L\,\mathrm{Re}^{-3/4}$, at which the viscosity terminates the kinetic energy cascade.

Notice that the two different scaling arguments described above have quite a different character. In the Blasius argument, for each $x$ there is one characteristic distance in the $y$-direction (set by $Y(x)$) and the structure of the flow is describable by that scale alone. In the Kolmogorov argument, there are a continuum of characteristic distances, described by the variable $b$. The same basic structure reappears at all length scales and, in fact, one expects that structures are nested inside of one another.

One can also describe the results of both arguments by kind of universality idea. Universality simply means that the solution to a given problem is, in an appropriate sense, independent of the details of the problem set-up. In the Blasius case, we neglected some terms in the hydrodynamic equations, because the answer was insensitive to these terms. In the Kolmogorov case, the reasoning is once again deeper. One argues that for each volume of side $b$, with $b$ in the appropriate scaling range, one can produce essentially the same kind of theory as one uses for the entire turbulent flow. That's scaling. But furthermore one argues that the only important characteristics of that volume are its average velocity and its dissipation rate, $\varepsilon$. That's universality. (See refs. 22 below.) We still do not know whether the Kolmogorov theory is essentially correct.

## 2. Scaling and universality at critical points

The same ideas have been applied quite successfully to the study of behavior near phase transitions. The essential scaling ideas were stated by B. Widom

[7]. Imagine that we have a system, like for example the Ising model, and that this system has a local fluctuating quantity $O(\boldsymbol{r})$. Here $O$ might be, for example, the magnetization or the energy density. The main point is that at criticality fluctuations appear at all length scales. Let $O_b(\boldsymbol{r})$ represent the result of averaging $O(\boldsymbol{r})$ over a volume of order $b^d$ – with $d$ being, of course, the dimensionality of the space. One can expect that this quantity might scale as some power of $b/a_0$, $a_0$ being a microscopic distance, say as $(b/a_0)^{-x}$. The critical index, $x$, will of course be different for different fluctuating quantities, $O$. There are two aspects of the theory which should be mentioned here. One is that any quantity $O$ can be expanded in a set of basic fluctuating quantities (or operators) which each have simple scaling properties. The sum converges rapidly for $(b/a_0) \gg 1$ because there are only a few operators with small values of $x$. Thus there is a natural classification and ordering of operators in the theory. The second aspect arises specifically because we are in a system described in equilibrium statistical mechanics. For each fluctuating quantity $O(r)$ there is a corresponding change in the quantity – variously called the free energy, Hamiltonian, or action – which generates the Gibbsian probability distribution. In units in which $kT = 1$, this change is

$$\Delta F = \sum_{\boldsymbol{r}} O(\boldsymbol{r})h \,. \tag{2.1}$$

Here $h$ is the thermodynamic quantity conjugate to $O$. In this context, it is generally called a ‘field’. For example if $O(\boldsymbol{r})$ is the magnetization density then $h$ is the magnetic field, while if $O$ is the energy density, $h$ is the temperature change. Since $\Delta F$ is a logarithm of a probability, it is unchanged under scaling transformations. Therefore, if the index of the operator $O$ is $x$, then the corresponding scaling index of the field $h$ is [8]

$$y = d - x \,. \tag{2.2}$$

The few fields with $y$ greater than zero play a special role in the theory. These fields are termed ‘relevant’. A perturbation including relevant terms will drive one away from the critical point. Conversely, perturbations including only fields with $y$ less than zero will have no effect on the critical behavior. Systems which have Hamiltonians which are different only in terms proportional to these irrelevant operators are said to lie in the same ‘Universality Class’. In this way, all critical behavior is classified and Hamiltonian arranged in equivalence classes. (There is an exceptional intermediate case in which there is a field with $y = 0$. Changes in the Hamiltonian proportional to such a marginal operator can lead to continuous variations in the universality class [9]. But most critical systems do not contain such a marginal operator.)

Near the critical point, these operators and indices can be used to derive scaling statements similar in character to those in eq. (1.2). For example, in a critical system of size $L$, a quantity $O_b$ with critical index $x$ would have fluctuations which obey

$$\langle O_b^2 \rangle = (b/a_0)^{-2x} C^*(b/L) . \tag{2.3}$$

Here $C^*(\eta)$ is a correlation function expressed in a coordinate, $\eta = b/L$, appropriate for this kind of 'finite sized scaling'. By using scaling relations like these one can estimate the sizes and length dependence of all kinds of quantities in the critical region.

A large variety of different workers contributed to the formulation of this scaling and universality picture during the period centered about 1965. I point with pride to a review paper of 1966 [10] in which a group of us at the University of Illinois surveyed *all* (!) the literature which dealt with experiment and theory near critical points and concluded that the results could very well be correlated by this picture [11].

A few years later, K.G. Wilson capped off this line of development by converting the essentially phenomenological considerations of the earlier period into a calculational tool via the method of the renormalization group [12]. The basic idea was to consider the Hamiltonian $H$ which generates the statistical weight. This Hamiltonian depends upon some basic set of variables. For the Ising model this is the local magnetization $\sigma(\boldsymbol{r})$. So the weight is $\exp -H\{\sigma\}$. Wilson then looked at what happens when one eliminates degrees of freedom, for example by eliminating short wavelength components from $\sigma(\boldsymbol{r})$. This change should result in a new statistical weight with a new form of the spin–spin coupling. By construction, new and old weights have the same consequences at the critical point. Wilson's key insight was that universality and scaling would be achieved if after many transformations the coupling structure of the weight approached a limit, a 'fixed point'. He showed how this fixed point could be calculated, and from it all aspects of critical behavior followed [13].

Wilson's work was performed in the context of field theory. In fact, the name 'renormalization group' had long been applied to related calculations in electrodynamics. Other work showed that this problem of critical phenomena was somehow very deeply connected to the Lagrangian field theories studied in particle physics. For example, Schultz, Mattis and Lieb [14] worked out the Onsager [15] solution of the two-dimensional Ising model as an example of a fermion field theory. The work of Kadanoff and Ceva [16] showed how scaling ideas implied that stress tensors and energy densities were natural parts of two-dimensional Ising model behavior. Polyakov [17] saw that the existence of the stress tensor in a rotationally symmetric situation implied that the critical field theory had a far richer symmetry than scaling. It is invariant under a larger group of transformations called the conformal group. In two dimensions this conformal symmetry is particularly rich and, in fact [18], it almost fully defines the possible critical behaviors.

The end result is that we understand critical phenomena. Each critical problem defines its own little world, described mostly by the dimensionality and symmetry of the underlying problem. As a result of universality the basic laws within each of these worlds are fixed and unchangeable. But because of fluctuations in the operators, things do happen. Each near-critical system supports elementary excitations which can be interpreted as massive elementary particles. These 'particles' scatter off of one another and behave as lively inhabitants of the universe in question. The operator classification ensures that the things that happen can be well described in terms of a known list of possibilities and measurements. Because of the field classification, there are a finite list of different significant ways of disturbing the world. In fact, the disturbances can be fully described by giving position-dependent values of the relevant fields. These fields play the same role as the velocity and dissipation rate in the Kolmogorov theory of turbulence. Because of scaling, the resulting geometrical structures are particularly simple. Basically the very same spatial structures are repeated again and again, at different places, in different magnifications, one inside the other. Thus, we find a beautiful toy world, almost fully understood, and correctly describing critical phenomena. Of course, the interesting question is whether other problems can be fitted into the same framework.

## 3. Chaos: scaling and universality

Mitchell Feigenbaum was responsible for a particularly elegant and important application and extension of the ideas described above. He was looking at dynamical systems, examining the patterns of $x$-values produced by the successive applications of mapping function, $G$. That is he was concerned with the character of the sequence $x_j$, $j = 1, 2, \ldots$, generated by the successive applications of the function, $G$, via

$$x_{j+1} = G(x_j) . \tag{3.1}$$

In one example, he examined the so-called logistic map in which $G(x) = rx(1-x)$, where $r$ is a parameter which could be varied to change the character of the $x$-sequence. He noted, following earlier authors [19], that as $r$ is increased from one the long term behavior of the sequence changes from a fixed point (i.e. an approach to a fixed value, $x^* = 1 - 1/r$), to a cycle of length two ($x$ approaching different values on even and odd values of $j$), to a cycle of length four, . . . to a cycle of length $2^n$. Then a specific value of $r$, called $r_\infty$, results in a cycle of length $2^\infty$. For higher $r$ yet, chaotic behavior ensues in which the sequence essentially never repeats itself. Feigenbaum observed two kinds of scaling: one that the length $2^n$ cycle first appears at an $r$-value, $r_n$, which obeys:

$$r_n = r_\infty - (\text{const})\delta^{-n} \tag{3.2}$$

in the limit of large $n$. The other scaling was a special behavior which occurred near the $x$-value for which the map is extremal ($x = \frac{1}{2}$ in the logistic map). If you started out at this value for $x_0$ then for large $n$

$$x_{2^n} = x_0 - (\text{const})\alpha^{-n} . \tag{3.3}$$

Here $\alpha$ and $\delta$ are two scaling indices which turn out to be 'universal'. In this case, universality means that these very same values of the critical indices arise from infinite period-doubling in a wide variety of different mapping problems. Building upon the analogy to critical phenomena, Feigenbaum [20] used the universality and scaling ideas to develop a renormalization theory for the period doubling. In this case, the theory was based upon the idea that the high order cycles (say of order $2^n$) could be realized as fixed points of the function, $G$, composed with itself $2^n$ times. Further, Feigenbaum noted that, at $r_\infty$, the large-$n$ result of this multiple functional composition (plus a scale change in $x$) was a fixed point function, $G^*$. Under iterations of this renormalization process, small deviations from the fixed point function grow with a discrete spectrum of eigenvalues. Once again, there are only a few relevant (growing) eigenvalues. Thus the renormalization group analysis was carried over to the new problem with the function (and its compositions) playing the role of the Hamiltonian while the $x_j$ sequence plays the role of the observable quantities.

There is a technical difference between the critical phenomena case and the one of period doubling: In the former, there is a duality between operators and changes in the free energy; in the latter, the connection between observables and $G$ has been broken. To get observables, imagine getting $2^n$ successive $x_j$ values and then forming the small differences:

$$\Delta_j = |X_{j+2^{n-1}} - X_j| . \tag{3.4}$$

It is observed that these differences have a whole range of critical indices.

Specifically $\Delta_j$ achieves an index-value [21] $\alpha_j$ by having

$$\Delta_j = (2^{-n})^{\alpha_j} . \tag{3.5a}$$

Define a weight for the critical index $\alpha$ by saying that $W(\alpha)\,\mathrm{d}\alpha$ is the number of times that $\alpha_j$ lies within an interval $\mathrm{d}\alpha$ about a given value $\alpha$. After a normalization $W(\alpha)$ is a kind of weight function for the critical index. In the limit as $n$ goes to finity, we can write this weight as

$$W(\alpha) = (2^n)^{f(\alpha)} . \tag{3.5b}$$

Evidently, $f(\alpha)$ serves as a kind of entropy function for the occurrence of the continuously varying critical index, $\alpha$. For many problems, including this period doubling, $f(\alpha)$ turns out to be independent of $n$ for large $n$. When this happens over an interval of $\alpha$, we say that we have a multifractal behavior [22].

Like the critical indices of eqs. (3.2) and (3.3), the function $f(\alpha)$ is universal. My experimental colleagues at the University of Chicago constructed a hydrodynamic situation in which a fluid heated from below was pushed to the onset of chaos. Then, the $f(\alpha)$ curves derived from two of the theoretically described 'routes to chaos' [23] were compared with the corresponding curves derived from the experiments [24]. The excellent agreement between the two clearly showed that the universality concept extended to the real experimental system.

In many ways, Feigenbaum's work was a very big surprise. In particular, the mathematicians who had studied chaos earlier noticed, quite correctly, that these problems were far too rich to be characterized within an ordered list of universality classes. Indeed, typically, these problems seem to be classifiable only with the aid of infinities of continuously varying parameters. But by focusing upon the onset of chaos, Feigenbaum had picked out a group of problems which could, once again, be divided into sets, universality classes and described via scaling and renormalization.

Despite this further success of the scaling/universality/renormalization approach, there are some hints that the critical phenomenon example will not be infinitely extendable. We know about dynamical systems problems which give scaling but no universality. One such example is the construction of Julia sets [25–27], which are mathematical structures somewhat analogous to chaotic attractors.

Nonetheless it is worthwhile to look for some remnant of universality in low-dimensional chaos, carried beyond onset. The analysis of Gunaratne, Jensen, and Procaccia [28] shows there is a kind of topological universality visible in the chaotic regime, but no real metric universality. More specifically, they look at mapping problems and focus upon the cycles, calculating their lengths and seeing the structure of very long cycles. They ask questions about which cycle is close to another, without asking 'how close?'. And they see that, if they look at the right kinds of problems, the set of all cycles is stable under appropriately chosen small changes in the mapping problem. They then hope to use this observation to obtain a classification of chaotic problems by working about structurally stable points in the phase space. Thus while they aim for universality, they have given up all elements of scaling. On the contrary, they conclude that in really chaotic problems there is no universality in distance measurements. This conclusion agrees with our experience of Julia sets. For myself, I worry whether there can be deep structure to a universality theory which leaves out all measurements of distance.

## 4. Dynamical systems with many degrees of freedom

### *4.1. Dynamic critical phenomena*

The next step is to look at dynamical systems with many degrees of freedom to see how much scaling and universality they show. The simplest case is dynamical critical phenomena in which the static correlations of near-critical behavior combine with conservation laws to engender interesting correlations in space and time [29]. There are well developed renormalization theories [30] based upon Lagrangians with extra fluctuating operators beyond those needed in the equilibrium theory [31]. Of course the extra operators depend upon both space and time. The resulting theory indicates universal behavior, with the universality depending upon both the equilibrium critical behavior and also the types of conservation laws. Thus, we have one more successful extension of the standard synthesis which I have been expounding.

### *4.2. Self-organized criticality*

But there do exist dynamical processes which produce scaling results without having any obvious underlying equilibrium critical behavior. When one gets to dynamical systems with many degrees of freedom, we no longer know the nature of the fundamental theory and so, for example, we cannot know whether or how renormalization group concepts apply. I discuss two examples here.

The first example is called DLA for diffusion limited aggregation. It is a model invented by Witten and Sander [32], in which a fractal [33] aggregate is grown by a step by step process. The aggregate sits on a lattice. A walker, added at infinity, undergoes a random walk process until it comes to a neighboring site to one already occupied. Then the walker stops and its final site is added to the aggregate. This whole process starts once again with another walker added at infinity. The aggregate thus produced is extremely tenuous and contains long branching arms with considerable space between the branches. Question: can this object be understood via concepts of scaling and universality? Is there somehow an underlying renormalization group description? Despite considerable study, it is fair to say that we simply do not know. It does appear that the aggregate itself is properly described as a fractal, but the measurement of its fractal dimension has been peculiarly difficult. Alternatively, one can describe the growth process by asking what is the probability that a new particle will be added at a given surface site. This probability varies over a wide range and is best described by a range of critical indices, in a multifractal formalism [34].

In DLA, the dynamics produces its own critical ordering. This kind of situation is termed by Bak, Tang and Wiesenfeld [35] to be one of "self-organized criticality". There are several other interesting examples in which the dynamical process produces an object which is marginally stable, and hence shows very long-ranged correlations [36]. One situation studied by Bak et al. and by others [37], is one in which model sand is added grain by grain to a model sandpile, built upon a regular $d$-dimensional lattice. In between additions, there are cascades of events in which sand falls downhill in response to a too-large local slope of the pile. These 'avalanches' can be small or they can cover the entire system many times over. Once again, it is interesting to ask about whether there is some scaling or universality. For example, one can study the nature of the probability distributions $\rho(X, L)$ for the probability that an event of size $X$ will occur in a system with spatial extent $L$. As before, Widom scaling is the statement

$$\rho(X, L) = L^{-\beta}\rho^*(X/L^{\nu}) \tag{4.1a}$$

while multifractal behavior is one in which

$$(\ln \rho(X, L))/\ln L = f(\alpha) \qquad \text{with } \alpha = (\ln X)/(\ln L) \tag{4.1b}$$

for $X$ and $L$ much bigger than one. Numerical work suggests that the multifractal behavior seems preferred in one dimension, while for two dimensions the question remains open. For both dimensionalities, the numerics indicates several universality classes, but universality within each class [38].

Up to now, we have had neither a completely convincing formalism nor a fully credible phenomenology for describing these essentially dynamical objects and processes. It is interesting to notice that they tend to be multifractal. Since none of the most conventional field theory/statistical mechanics systems seem to have a continuum of critical indices, we might wish to question whether these dynamical systems are really described by a simple field theory. This question is an important one in that many of our methods of setting up and thinking about such problems are closely based in Lagrangian field theory. But, perhaps these problems do have a Lagrangian someplace after all. One does know problems (e.g. percolation and random resistor networks) in which the behavior is multifractal but the system is understandable as a limiting case of one with a Lagrangian description. Alternatively, they could be describable by a Lagrangian which has no symmetry between space and time. However, notice that in both DLA and sand slides, the step between elementary addition events involves complex processes: either an entire random walk or a whole avalanche. Thus, the elementary processes in this proposed Lagrangian are far from simple. For this reason, the Lagrangian formulation may fail. My own hope is that sand slides and DLA and many other dynamics problems are essentially new and have some – as yet unknown – formation, combining some elements of scaling, universality, and renormalization group.

### 4.3. *The binomial distribution: a simple multifractal example*

However, one can set up renormalization examples which do have a multifractal character. I do this here. Since the work of Billingsley [39], it has been known that the binomial distribution provides an example of a probability which has a rich asymptotic structure and is, in modern terms, multifractal. Consider $Q$ objects which are distributed randomly between 2 bins. The total number of ways that this can be done is $L = 2^Q$. (If we visualize adding the objects one at a time then $Q$ can be considered to be an analog of a temporal variable.) The probability that $P$ objects will show up in the first bin is, of course,

$$\rho(P, Q) = Q!/(P!(Q-P)!2^Q) . \tag{4.2}$$

Let $P, Q \gg 1$. From the Stirling approximation for factorials we then find that $\rho(P, Q)$ has a multifractal form with

$$f(\alpha) = -\ln 2 - \alpha \ln \alpha - (1-\alpha)\ln(1-\alpha) \tag{4.3}$$

and $\alpha = P/Q$. Near the peak (at $\alpha = \frac{1}{2}$) the distribution is Gaussian, but the result (4.3) also works far into the wings of the distribution.

To derive a renormalization method for computing $f(\alpha)$ recall Pascal's Triangle for binomial coefficients which is the statement

$$\rho(P, Q) = [\rho(P-1, Q-1) + \rho(P, Q-1)]/2 . \quad (4.4)$$

Now let us make the change of variables:

$$f(\alpha, Q) = \frac{\ln \rho(P, Q)}{Q} ,$$

substitute into eq. (4.4), and expand in $1/P$ and $1/Q$ to find the simple renormalization equation

$$Q \frac{\partial f}{\partial Q} = \ln\left\{\exp\left[-f - (1-\alpha)\frac{\partial f}{\partial \alpha}\right] + \exp\left[-f + \alpha \frac{\partial f}{\partial \alpha}\right]\right\} - \ln 2 . \quad (4.5)$$

The fixed point solution makes the right hand side of (4.5) vanish. The $f(\alpha)$ of eq. (4.3) is a correct solution to this equation in the limit as $q$ goes to infinity.

Can one attack the sandslide problem in this fashion? I cannot yet see how to achieve this but I would dearly love to do so. But, notice that the calculation I have just done is an essentially traditional application of real space renormalization methods. If, as I hope, there is something really new in DLA and sand slides then no progress is likely without a new idea, as yet unforeseen.

The systems with novel scaling states based upon self-organized criticality are going to be an interesting subject of study during the next few years. Many workers have the tantalizing feeling that solutions to one or another of these problems is just within reach, but so far they seem to have eluded us.

### *4.4. Back to turbulence*

Fully turbulent systems, which are even more interesting than the ones with chaos or those with self-organized criticality, have proven very hard. One hopes, nevertheless, that some important simplification will arise because of the very large number of excited degrees of freedom. Experiment shows some suggestion of multifractal behavior [40], and other suggestions that simple scaling laws might hold [41]. Apparently similar spatial structure reappear in a wide variety of problems [42]. In that weak sense, there is some universality in turbulence. But, we cannot be really sure about how many parameters (fields) we need to define what is going on in a given turbulent region. Can we get by with a finite number? Is Kolmogorov's four ($u_b$ and the dissipation rate $\varepsilon$) the right number? Are there structures inside of structures? Which are the really instructive situations: a randomly stirred system [43], or one in which turbulence is decaying in time [44] or one in which turbulence is spreading in space, or one in which there is in continual non-random forcing? How does one begin to attack this kind of problem?

If one asks how much universality and scaling really hold for a turbulent system, the only possible answer is 'nobody knows'.

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