Part 3: Lattice: Quantum to Ising to RG

From Classical Stat Mech to Quantum to RG

All of quantum mechanics on one slide

To do quantum mechanics, one starts with a complete set of states $|q)$ and $(p|$ which have the ortho-normality property $(q|q') = \delta_{q,q'}$ and a completeness relation

$$\sum_q |q)(q| = 1 \tag{4.1}$$

and a trace operation

$$\text{trace } \mathcal{P} = \sum_q (q|\mathcal{P}|q) \tag{4.2}$$

Heisenberg representation $P(t)=e^{-iHt}\, P\, e^{iHt}$. Let $T(t) = e^{-iHt}$

Partition Function $Z(\beta) =$ trace $T(-i\beta)=\sum_\alpha \exp(-\beta\epsilon_\alpha)$

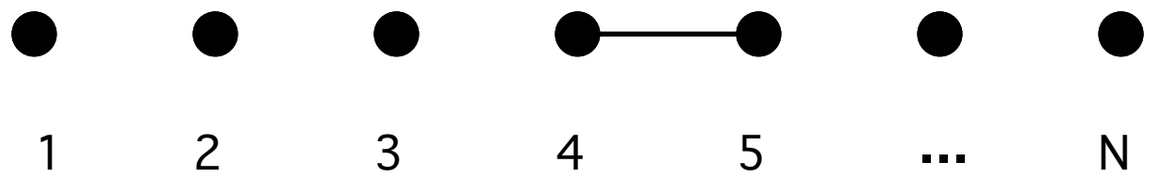
Average $\langle Q\rangle = [\text{trace}\, T(-i\beta)\, Q]/\, Z(\beta)$

Two Times $\langle Q(s)\, P(t)\rangle = [\text{trace}\, T(-i\beta)\, Q(s)\, P(t)\,]/\, Z(\beta)$

For grand canonical ensemble use $T(t)= exp[-i(H-\mu N)t]$

Nearest Neighbor Interactions in Stat Mech on a One-Dimensional Lattice

Imagine that we wished to understand a problem involving nearest neighbor interactions one a one dimensional lattice which looks like



The variables at the different lattice sites are $q_1, q_2, \ldots\ldots, q_N$. The statistical weight is a product of terms depending on variables at the nearest neighboring lattice sites of the form $w(q_j,q_{j+1})$ so that the entire calculation of the partition function is*

$$Z = \sum_{q_1,q_2,\cdots,q_N} exp[w(q_1,q_2) + w(q_2,q_3) + \cdots + w(q_{N-1},q_N) + w(q_N,q_1)] \qquad \text{iii.1}$$

Notice that we have tied the two ends of the lattice to one another via a coupling $w(q_N,q_1)$. We have essentially used periodic boundary conditions. This calculation can be converted into a quantum mechanics calculation using a quantum mechanical operator, *T*, defined by its matrix elements

$\langle q|T|p\rangle = exp[w(q,p)]$

Now substitute this expression into the partition function calculation. We then have,

*We shall not describe the nearest neighbor problem in terms of a statistical mechanical Hamiltonian since we are saving the word "Hamiltonian" for the quantum problem which will replace it.

From Classical to Quantum:

$$Z = \sum_{q_1,q_2,\cdots,q_N} < q_1|T|q_2 >< q_2|T|q_3 > \cdots < q_{N-1}|T|q_N >< q_N|T|q_1 > \qquad \text{iii.2}$$

If you recall the definition for matrix multiplication,

$$< q_1|TS|q_3 >= \sum_{q_2} < q_1|T|q_2 >< q_2|S|q_3 >$$

you will see at once that the partition function is $Z = \sum_{q_1} < q_1|T^N|q_1 >$

so that Z= trace T^N

In order to get something familiar, imagine that T is an exponential of a Hamiltonian, specifically T=exp(−τH)), where H is a Hamiltonian defined in terms of w. In terms of matrix elements

$<q|T|p>= \quad exp[w(q,p)]= <q|\exp(-\tau H)|p>$

In fact, T is what we called before $T(-i\tau)$. If we write the trace in terms of the eigenvalues, ϵ_α, of H we have

so that Z= trace $T(-i\tau)^N$ =trace $e^{-N\tau H}= \sum_\alpha \exp(-N\tau\epsilon_\alpha)$ iii.3

Summary

$$Z = \mathrm{trace}_{q_1}\mathrm{trace}_{q_2}\ldots\mathrm{trace}_{q_n}\prod_{j=1}^{N}\exp(w(q_j,q_{j+1})) \qquad (4.10)$$

or in a more compact notation

$$Z = \mathrm{Trace}_{\{q\}}\exp[W\{q\}] = \mathrm{trace}\exp[-H\tau N] \qquad \text{iii.4}$$

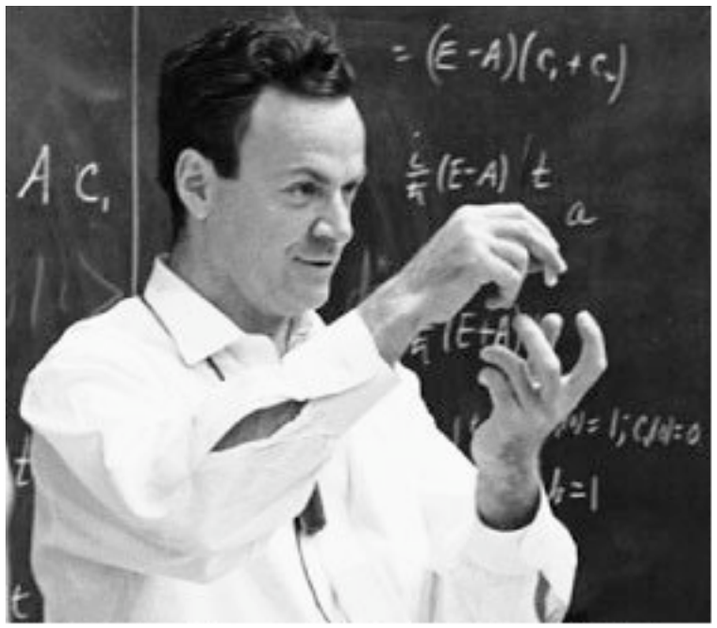
Note that we use the word "trace" to represent both a quantum and a statistical mechanical sum. The trace in equation 4.10 is a statistical sum. The first Trace in equation iii.4 is a statistical mechanical sum, the second is a quantum mechanical trace. We use a capital "T" when we sum over many variables and a lower case one when we sum over only one or a few.

The point of the argument is that they have a direct translation into one another: Every quantum mechanical trace can be converted into a one-dimensional statistical mechanics summation and *vice versa*.

Feynman showed how to convert problem of quantum mechanics into a path integral. We have essentially put his path on a one-dimensional lattice.

Every quantum mechanical trace can be converted into a one-dimensional statistical mechanics problem and *vice versa*. More generally, d dimensional quantum mechanics converts to d+1 dimensional stat mech. (Here, 0 dimensions of quantum becomes 1 of statistical mechanics).

The basic idea about going up and back between the two disciplines is due to Feynman, and his invention of the path integral. As far as I know, Feynman never quite said the sentence written in blue. The point was pursued and made explicitly by Kenneth Wilson, and used in his invention of the modern renormalization group. I'll come back with more about that later.



Richard Feynman

Feynman's case: particle mechanics

The simplest and most fundamental problem in quantum theory is a particle in a one-dimensional potential, $H=p^2/(2m)+U(q)$, where p and q obey $[p,q]=-i\hbar$. However, we shall stick with units in which $\hbar=1$.

We assert, without proof, that the exponential of this operator has the matrix element

$$\langle q|T(-i\tau)|q'\rangle=\langle q|\exp[-\tau H]|q'\rangle = \exp[-m(q-q')/(2\tau)-\tau U(q)] \qquad \text{iii.5}$$

for small values of τ. Because τ is small, q and q′ are necessarily close to one another. For that reason, we can replace U(q) by U(q′) or by [U(q)+ U(q′)]/2 in the analysis that follows. (These choices are close to equivalent, but they are not the same because p and q do not commute.)

Calculate the matrix element of $\exp[-\tau p^2/(2m)]$ between position eigenstates.

Imagine that we wished to know the eigenvalues of the Hamiltonian, *H*. We could, for example, numerically calculate the integral of products of matrix elements as given above. As the number of steps times τ goes to infinity we would pick out the lowest eigenavlue as the leading term in

$$\text{trace}\, T(i\tau)^N = \text{trace}\; e^{-N\tau H} = \sum_\alpha e^{-N\tau\varepsilon_\alpha} \qquad \text{iii.6}$$

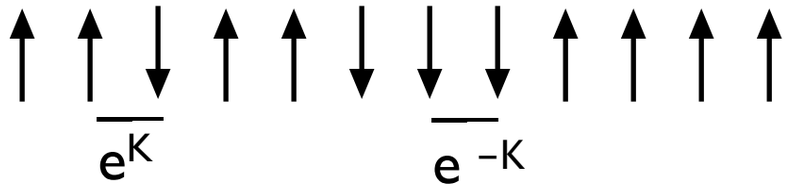
This approach provides a powerful method for both numerical and analytic approaches to quantum problems. However, I'll do a simpler case here.

The Ising Linear Chain

We calculate the partition function in the simplest case of this kind. Take an Ising model with spins σ_j at sites $j = 1, 2, \ldots, N$. Take the magnetic field to be zero and arrange the couplings so that immediately neighboring sites (j and $(j+1)$ have a coupling K. The statistical weight for two neighboring sites having spin-values σ and σ' is then defined to be

$$e^{w(\sigma,\sigma')} = e^{K\sigma\sigma'} = (\sigma|\mathcal{T}|\sigma') \tag{4.17}$$

linear chain



This kind of two by two system is generally analyzed in terms of the Pauli matrices which are the four basic matrices that we can use to span this two by two space. They are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.18}$$

In going up and back between the notation of equations (4.17) and (4.18) we have to think a little. In (4.17), we interpret σ and σ' as eigenvalues of the matrix τ_3. Any two by two matrix, M, can be written in terms of the eigenstates corresponding to these eigenvalues:

$$M = \begin{pmatrix} (-1|M|-1) & (-1|M|1) \\ (1|M|-1) & (1|M|1) \end{pmatrix}.$$

The transfer Matrix

A useful form for these Pauli matrices is

$$(\sigma|\mathbf{1}|\sigma') = \delta_{\sigma,\sigma'} \qquad (\sigma|\tau_1|\sigma') = \delta_{\sigma,-\sigma'} \qquad (4.19b)$$
$$(\sigma|\tau_2|\sigma') = i\sigma\delta_{\sigma,-\sigma'} \qquad (\sigma|\tau_3|\sigma') = \sigma\delta_{\sigma,\sigma'}$$

these matrices have a very direct physical meaning.
The matrix τ_3 is diagonal in the σ-representation and represents the spin. Conversely, τ_1 has only off-diagonal elements. It is an operator whose effect is to change the σ-value.

The matrix element of the transfer matrix, *T,* is equal to e^K when σ = σ' and equal to e^{-K} otherwise. In symbols,

$$(\sigma|\mathcal{T}|\sigma') = \frac{1+\sigma\sigma'}{2}e^K + \frac{1-\sigma\sigma'}{2}e^{-K} \qquad (4.20)$$
$$= e^K \mathbf{1} + e^{-K}\boldsymbol{\tau_1}$$

Here the matrices in **bold** are the ones defined in eq. 4.19b. We can also write the result as an exponential, *T*=exp(-*H*) where

Dual Couplings

$$-\mathcal{H} = \tilde{K}_0 \mathbf{1} + \tilde{K} \tau_1 \qquad (4.22)$$

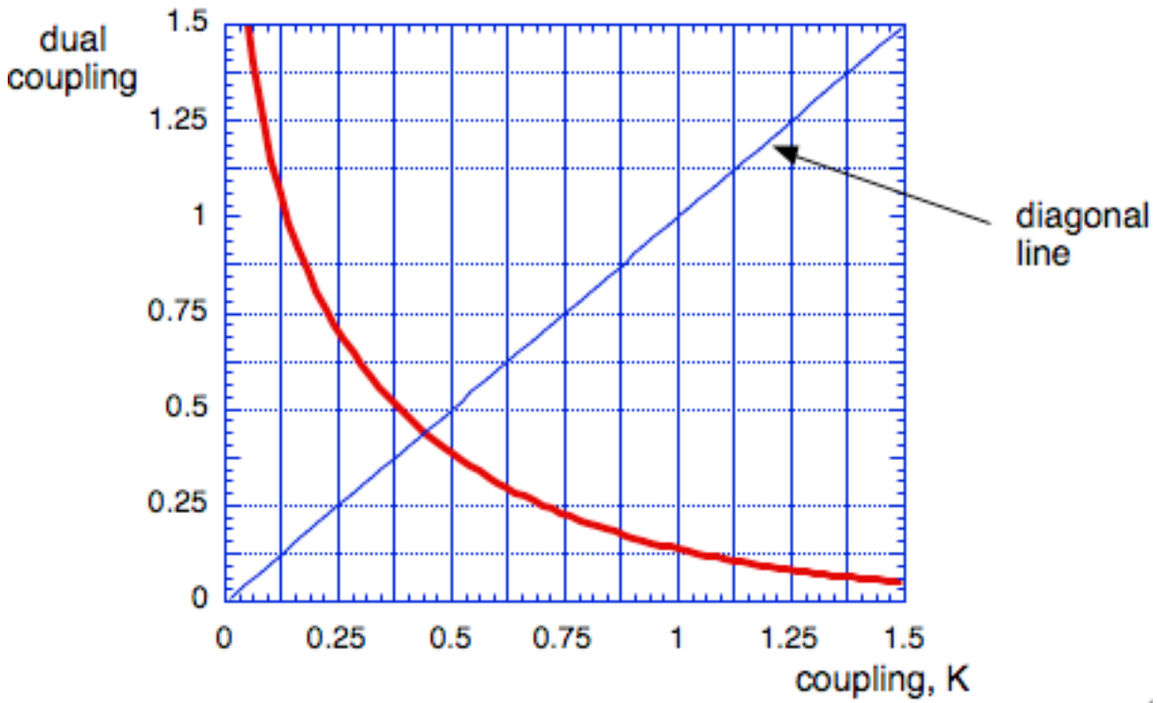
The quantity $\tilde{K}$ is said to be the dual of K. For a simpler notation, we call this function by another name so that the dual of K is D(K) This name implies in part that the function D(K) has the property that if it is applied twice that you get precisely the same thing once more:

D(D(K))=K or $D^{-1}(K)=D(K)$

How would I find the function D(K)?

$\tilde{K}$= D(K)=[ln (tanh K)]/2 $\tilde{K}_0$= [ln(sinh 2K)]/2

This function has the property that when K is strong its dual is weak and *vice versa*. This property has proven to be very important in both statistical physics and particle physics. Often we know both a basic model and its dual. Often models are hard to solve in strong coupling. But the dual models have weak coupling when the basic model has strong coupling. So then we get an indirect solution of the basic model.



Solution of the one-dimensional Ising model

From equation 4.20, we find that the partition function of the one-dimensional Ising model is

$$Z = \text{ trace } (e^K \mathbf{1} + e^{-K} \tau_1)^N$$

But the trace is a sum over eigenvalues and the eigenvalues of τ_1 are plus or minus one. Thus, the answer is:

$$Z = (2\cosh K)^N + (2\sinh K)^N \qquad (4.25)$$

If N is very large, the first term is much larger than the second and thus in this limit of large system size:

$$-\beta F = \ln Z = N \ln(2\cosh K) \qquad (4.26)$$

What quantum mechanics problem have we solved?

More about quantum from the Long Chain

We should be able to say more about quantum problems based upon the analysis of the long chain. For example let us imagine that we wish to calculate the average of some quantum operator, *X(q)*, which happens to be diagonal in the q-representation. The text book goes through a long song and dance to prove a rather obvious result. You have seen that the trace in equation 4.10 pushes us into a sum over energy states, and if *N* is very large that sum reduces to a projection onto the ground state of the system. Specifically,

$$Z = \mathrm{trace}_{q_1} \mathrm{trace}_{q_2} \ldots \mathrm{trace}_{q_n} \prod_{j=1}^{N} \exp(w(q_j, q_{j+1})) \qquad (4.10)$$

becomes $Z=\exp(-\tau\epsilon_0)$

So if we insert an X, for any any operator *X*, in that sum the result should give what happens to that X in the ground state, specifically

$$(1/Z)\ \mathrm{Trace}_{\{q\}}\ \exp[W\{q\}]\ X = \langle 0|\ X(q)\ |0\rangle$$

In this way, we can use statistical mechanics to calculate the average of any operator in the ground state. If we do not take *N* to infinity, we can do the corresponding calculation to calculate the average of any operator at a inverse temperature ($\beta-$ value) equal to N τ.

By playing with the times in an appropriate fashion, we can even calculate time-dependent correlation functions in the ground state or in a finite-temperature state.

Statistical Correlations in a Long Chain

We should be able lots about the statistical mechanics of a the long chain with Ising style interactions. For example, let us calculate the average of the jth spin on a long chain or the correlations among the spins in the chain. Start from

$$Z = \mathrm{Tr}\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j\ ]$$

$$< \sigma_k >= (1/Z)\mathrm{Tr}\,\sigma_k\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j\ ]$$

$$< \sigma_k\sigma_{k+r} >= (1/Z)\mathrm{Tr}\,\sigma_k\sigma_{k+r}\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j\ ]$$

Here Tr means "sum over all the N spin-values". We use periodic boundary conditions. In this equation all the σ's are numbers, and they commute with each other.

We can make the calculation easier by replacing all the couplings by their expressions in terms of Pauli spins matrices giving these three calculations as, first,

$$Z=\mathrm{trace}_\tau \prod_{j=1}^{N} \exp\,(\tilde{K}_0 + \tilde{K}\tau_1) = \mathrm{trace}_\tau \exp[N\,(\tilde{K}_0 + \tilde{K}\tau_1)]$$

$$= (2\cosh K)^N + (2\sinh K)^N \qquad \approx \quad (2\cosh K)^N$$

The ≈ is an approximate equality which holds for large *N*. Note that in this limit the term with eigenvalue of $\tau_1 = 1$ dominates because the dual coupling is positive.

Average magnetization in a Long Chain

We know the answer: the system has full symmetry between spin up and spin down so that the average magnetization must be zero. Nonetheless, let's calculate

$Z\langle\sigma_j\rangle = \text{trace}_\tau \{\exp [-H\,\tau(j-1)]\ \tau_3\ \exp [-H\,\tau(N-j+1)] \}$

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$= \text{trace}_\tau \{ \tau_3\ \exp [-H\,\tau N] \} = \text{trace}_\tau \{ \tau_3\ \exp [-(K_0 + K\,\tau_1\,N] \}$

To evaluate the last expression we must take diagonal matrix elements of τ_3 between eigenstates of τ_1. Both such matrix elements are zero. why? Because τ_3 acts to change the value of τ_1 so that $\tau_3\ |\tau_1=1\rangle = |\tau_1=-1\rangle$ so that $\langle\tau_1=1|\ \tau_3\ |\tau_1=1\rangle = \langle\tau_1=1|\ |\tau_1=-1\rangle=0$. Therefore the entire result is zero and the average has the value zero, as expected.

$$\langle\sigma_j\rangle = 0$$

At zero magnetic field, the magnetization of the one-dimensional Ising model is zero. Thus, this Ising model has no ordered state. In fact no one-dimensional system with finite interactions has one. This model is always in the disordered phase at all finite temperatures.

Correlations in Large N limit

Let N be large. Z simplifies to $Z = \exp(N\tilde{K}_0 + N\tilde{K})$ since the τ_1=1 term dominates the trace

We start from*

$$Z \langle\sigma_j\sigma_{j+r}\rangle = \text{trace}_\tau \{e^{-(j-1)H}\ \tau_3\ e^{-rH}\ \tau_3\ e^{-(N-j-r+1)H}\},\qquad \text{for large } N$$

* Note how the ordering in space converts into an ordering in time.

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$$(\text{trace}_\tau\ e^{-NH})\ \langle\sigma_j\sigma_{j+r}\rangle = \text{trace}_\tau \{e^{-(N-r)H}\ \tau_3\ e^{-rH}\ \tau_3\},\qquad \text{so that}$$

The K_0 term is the same on both sides of the equation. It cancels.
For large N, the τ_1 =1 term dominates both traces. Since the effect of τ_3 is to change the eigenvalue of τ_1 this result is

$$\exp[\tilde{K}N]\ \langle\sigma_j\sigma_{j+r}\rangle \approx \{\exp[\tilde{K}N]\ \exp[-2\tilde{K}r]\},\qquad \text{for large } N \qquad \text{Consequently}$$

$$\langle\sigma_j\sigma_{j+r}\rangle \approx \exp[-2\tilde{K}r],\qquad \text{for large } N \qquad \text{iii. 7}$$

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ , being the distance between lattice points (usually called *a*) times 1/(2D(K))=1/($2\tilde{K}$).

Correlation Length

$$< \sigma_j \sigma_{j+r} >= exp(-2r\tilde{K}) \qquad = \exp(-ar/\xi)$$

Here *ar* is distance between the sites of the two spins.

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ, being the distance between lattice points (usually called *a*) times 1/(2D(K))=1/(2$\tilde{K}$). This falloff distance is very important in field theory, particle physics, and phase transition theory. In the latter context it is called the coherence length. It is also called the Yukawa distance because it first came up in Hideki Yukjawa's description of mesons. Here, in the one dimensional Ising model, we have a very large coherence length for large K. Specifically

$$1/(2\tilde{K}) = \xi/a \to \exp(2K)/2 \text{ as } K \to \infty$$

while is very small in the opposite limit of small K.

$$\xi/a \to 1/(-\ln(2K)) \text{ as } K \to 0$$

Hideki Yukawa

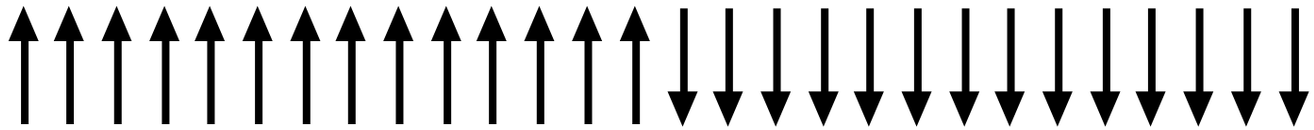
Large correlation lengths, or equivalently small masses, play an important role in statistical and particle physics since they indicate a near-by phase transition or change in behavior.

ξ = correlation length= a/[2D(K)] =a/(2$\tilde{K}$)

Bloch Walls in 1 d

In the Ising model at large values of the coupling, K, the spins tend to line up.

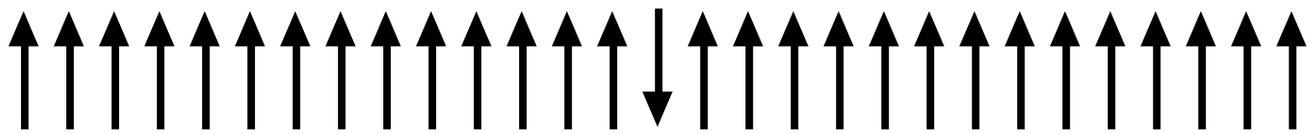
However, with a cost in probability exp(-2K) a whole region might flip its spins, producing a defect called a Bloch wall



This kind of defect produces the decay of correlations in the Ising model at low temperatures. In any long Ising chain, many such defects will be randomly placed and ruin any possibility of correlations over infinitely long distances.

This is the simplest example of what is called a topological excitation, a defect which breaks the ordering in the system by separating two regions with different kinds of order. Since ordering is crucial in many situations, so are topological excitations.

Notice that, at low temperatures, this kind of excitation is much more likely than a simple flip of a single spin. The wall costs a factor of exp(-K); the flip costs exp(-2K).



Renormalization for 1D ising,

following ideas of Kenneth Wilson, this calculation is due to David Nelson and myself

$$Z = \sum_{\sigma_1,\sigma_2,\cdots} \exp(W_K\{\sigma\}) = \sum_{\sigma_1,\sigma_2,\cdots} \exp(K\sigma_1\sigma_2 + K\sigma_2\sigma_3 + \cdots)$$

Rearrange calculation: Rename spins separated by two lattice sites: let $\mu_1=\sigma_1$; $\mu_2=\sigma_3$, $\mu_3=\sigma_5$, ...; and sum over every other spin, σ_2 , σ_4 .....

$$Z = \sum_{\mu_1,\mu_2,\cdots} \sum_{\sigma_2,\sigma_4,\cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1,\mu_2,\cdots} \exp(w'\{\mu\})$$

Note that sum over σ_2 , σ_4 ..... generates only nearest neighbor interactions for the μ's

$w'\{\mu\}$= const + K' μ_1 μ_2 + K' μ_2 μ_3 + ...

K' describes same system as before, with a new separation between lattice sites, which is twice as big as the old separation. Since the physical system is the same, physical quantities like the correlation length and the entropy are unchanged, but their description in terms of couplings and lattice constants has changed. In particular, the new lattice spacing is $a' = 2a$, but the correlation length is exactly the same $\xi' = \xi$. Since we know that the correlation length is given by

$\xi = a/[2D(K)]$, we know that the new coupling obeys $a/[2D(K)] = a'/[2D(K')]$ we find that the new coupling obeys $D(K') = 2\,D(K)$ before we do any detailed renormalization calculations. Since D is a decreasing function of K we know that the new, renormalized, coupling is smaller than the old one.

Renormalization Calculation

$$Z = \sum_{\mu_1,\mu_2,\cdots} \sum_{\sigma_2,\sigma_4,\cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1,\mu_2,\cdots} \exp(w'\{\mu\})$$

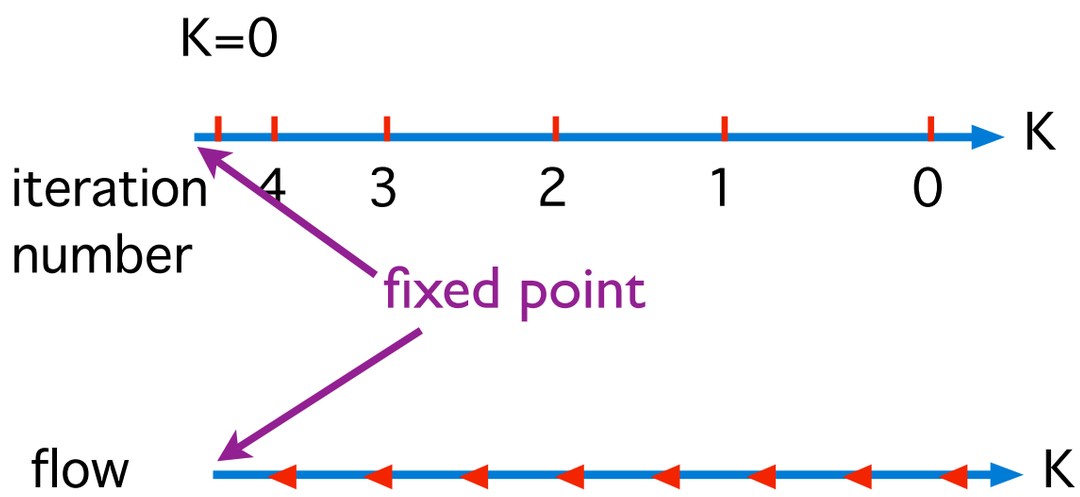
So the new nearest neighbor coupling term is given by

$$\exp(K_0' + K'\mu_1\mu_2) = \sum_{\sigma_2} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2)$$

which then gives us exp(K_0' +K')= e^{2K} +e^{-2K} and exp(K_0' – K')= 2
so that $e^{2K'}$ = cosh 2K.

The renormalization calculation tells us what we know already, namely that the one-dimensional model has no phase transition. A phase transition is a change in the long-ranged structure of correlations in a system. Here the couplings gradually weaken as you renormalize to longer and longer distances. All possible values of the coupling reduce to weak couplings at long distances. The system is always in the weak coupling phase. So there is no phase transition.

After many iterations coupling approaches fixed point at *K=0*



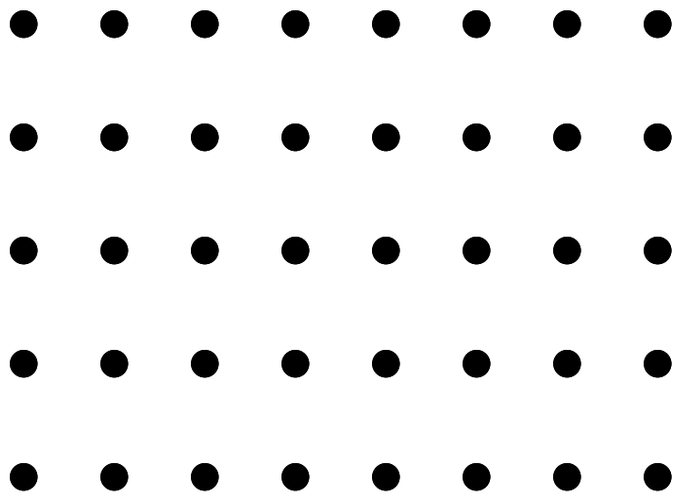
Ising Model in d=2

$$-H/(kT) = K\sum_{nn}\sigma_r\sigma_s + h\sum_r \sigma_r$$

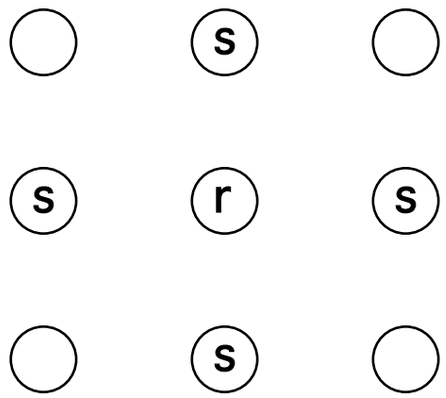
$\sigma_r=\pm 1$

nn indicates a sum over nearest neighboirs

square lattice



Onsager calculated partition function and phase transition for this situation

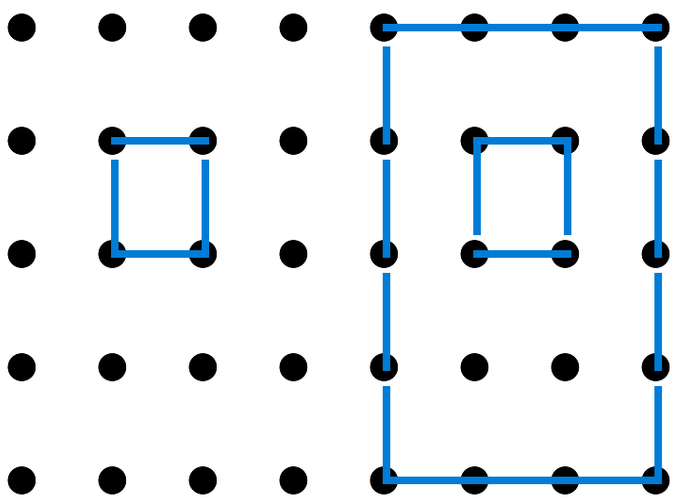


Nearest neighbor structure
s's are nearest neighbors to r
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors

High Temperature Expansion

Nearest neighbor structure
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors
Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

Z=(2 cosh K cosh K)^N < products of [1+ $\sigma\sigma'$ tanh K] >
= (2 cosh K cosh K)^N sum < products of (tanh K)^M >
for nonzero terms, when there are N sites



To get a non-zero value each spin must appear on a even number of bonds. You then get the lattice covered by closed polygons.

With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

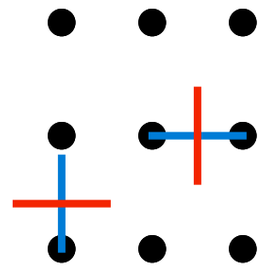
Low Temperature Expansion

Nearest neighbor structure

Bonds=$\exp(K\sigma\sigma') = e^{K}\delta_{\sigma,\sigma'} + e^{-K}\delta_{\sigma,-\sigma'}$

Bond $= e^{K}[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$

We draw these bonds differently from the high T bonds. We draw them rotated 90 degrees in comparison to the other bonds.

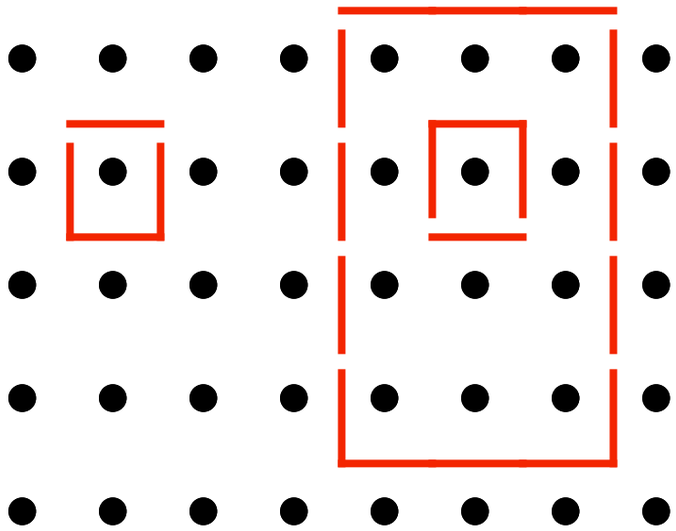


note e^{-2K}
$= \tanh \tilde{K}$

$Z=2(e^{K})^{N} <$ products of $[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}] >$

$= 2e^{NK}$ sum < products of $(e^{-2K})^{M}$ >

for nonzero terms



To get a non-zero term, assign a value to one spin. Then every time you cross a red line, change the spin-value to the opposite. Your valid pictures become a series of closed red polygons.

With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

Duality Hendrik Kramers and Gregory Wannier

Since the two expressions both give Z we get a relationship beteen a high temperature theory of Z and a low temperature one. We write our sum of products as exp[Nf(.)] where the . can be either exp(-2K) or tanh K depending on which expansion we are going to use. We then have

$$\ln Z = N[K] + N f[\exp(-2K)] = N \ln [\, 2 \cosh K \cosh K] + N f[\tanh K]$$

Let us assume that there is only one singularity in ln Z as K goes through the interval between zero and infinity. Since tanh K is an increasing function of K and exp(-2K) is a decreasing function of K, the singularity must be at the point where the two things are equal

$\tanh K_c = \exp(-2K_c)$.

After a little algebra we get $\sinh 2K_c = 1$

which is the criticality condition for two-dimensional Ising model. This criticality condition was later verified by Onsager's exact solution of the 2d ising model.

Further we might notice that ln Z must have a form of singularity in which the singular part of the partition function is even about this point.

Specific Heat = d^2 ln Z /d T^2

Further we might notice that ln Z must have a form of singularity which is even about the critical value of the coupling.

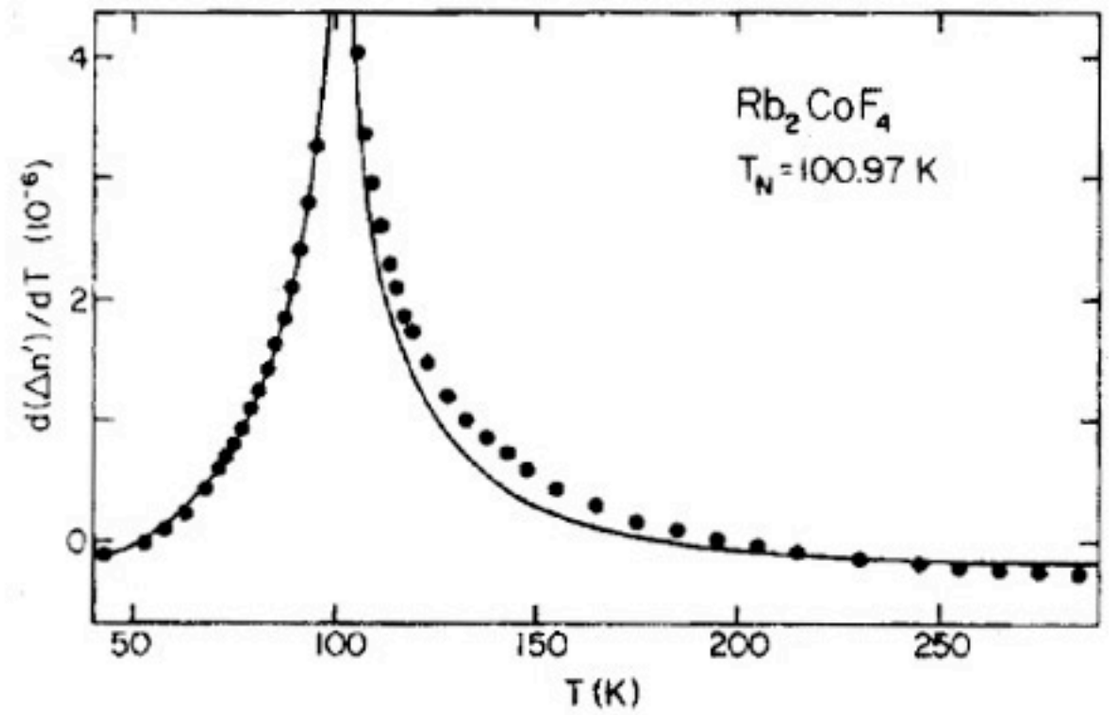


Figure 7. Variation of the magnetic specific heat, as a function of temperature for Rb_2CoF_4. The solid points (·) are experimental results of optical birefringence measurements shown previously to be proportional to the magnetic specific heat. The solid line is the exact Onsager solution for the two-dimensional Ising model with amplitude and critical temperature adjusted to fit the data, and a small constant background term subtracted. After Ref. 22.

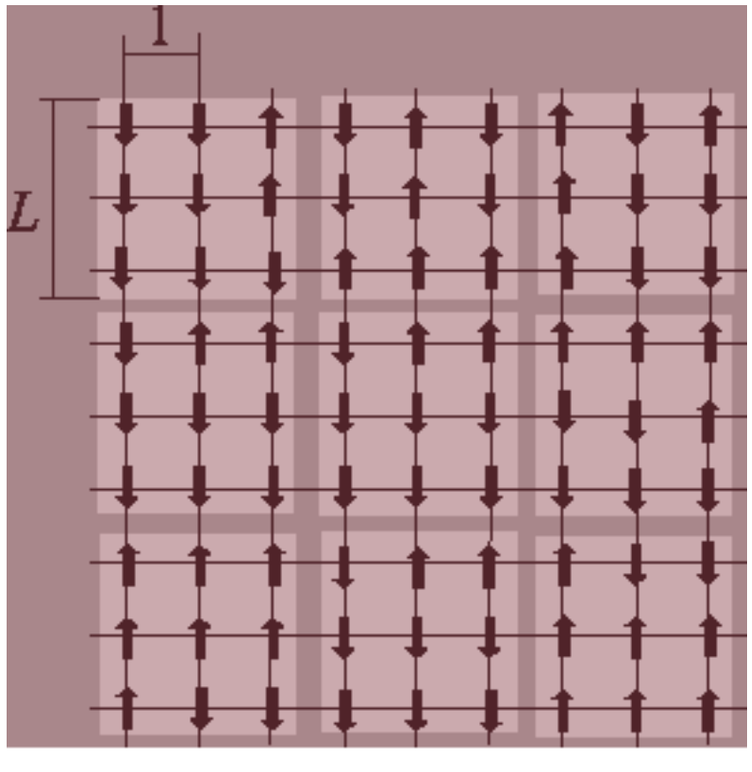
22. P. Nordblad, D.P. Belanger, A.R. King, V. Jaccarino and H. Ikeda., Phys. Rev. B 28. 278 (1983).

Renormalization for d-2 Ising model

A. Pokrovskii & A. Patashinskii, Ben Widom, myself, Kenneth Wilson.

$Z=\mathrm{Trace}_{\{\sigma\}} \exp(W_K\{\sigma\})$

Imagine that each box in the picture has in it a variable called $\mu_{\mathbf{R}}$, where the **R**'s are a set of new lattice sites with nearest neighbor separation 3*a*. Each new variable is tied to an old ones via a renormalization matrix $G\{\mu, \sigma\}= \prod_{\mathbf{R}} g(\mu_{\mathbf{R}},\{\sigma\})$ where g couples the $\mu_{\mathbf{R}}$ to the



fewer degrees of freedom produces "block renormalization"

σ's in the corresponding box. We take each $\mu_{\mathbf{R}}$ to be ±1 and define g so that, $\sum_{\mu} g(\mu,\{s\}) =1$. For example, μ might be defined to be an Ising variable with the same sign as the sum of σ's in its box. Now we are ready. Define

$\exp(W'\{\mu\})= \mathrm{Trace}_{\{\sigma\}} G\{\mu, \sigma\} \exp(W_K\{\sigma\})$

$Z=\mathrm{Trace}_{\{\mu\}} \exp(W'\{\mu\})$

If we could ask our fairy god-mother what we wished for now it would be that we came back to the same problem as we had at the beginning: $W'\{\mu\}=W_{K'}\{\mu\}$

Renormalization: $a \dashrightarrow 3a = a'$ $W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $Z' = Z$ $K' = R(K)$

Scale Invariance at the critical point: $\dashrightarrow$ $K_c = R(K_c)$

Temperature Deviation: $K = K_c + t$ $K' = K_c + t'$

if $t=0$ then $t'=0$

ordered region ($t>0$) goes into ordered region ($t'>0$)

disordered region goes into disordered region

if t is small, $t' = bt$. $b = (a'/a)^x$ defines x. b can be found through a numerical calculation.

coherence length: $\xi = \xi_0\, a\, t^{-\nu}$ 2d Ising has $\nu=1$; 3d has $\nu \approx 0.64$....

$\xi = \xi'$ $\xi_0\, a\, t^{-\nu} = \xi_0\, a'\, (t')^{-\nu}$

so $\nu = 1/x$

number of lattice sites: $N = \Omega/a^d$ $N' = \Omega/a'^d$

$N'/N = a^d / a'^d = (a'/a)^{-d}$

Free energy: F = non-singular terms $+ N f_c(t) = F'$ = non-singular terms $+ N' f_c(t')$

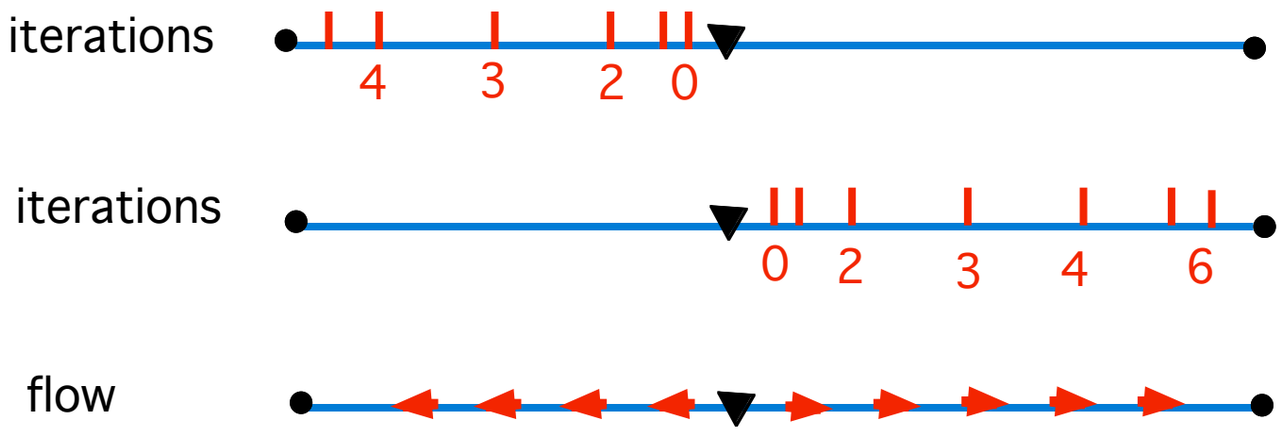
$f_c(t) = f_c^0\, t^{dx}$

Specific heat: $C = d^2F/dt^2 \sim t^{dx-2}$ form of singularity determined by x

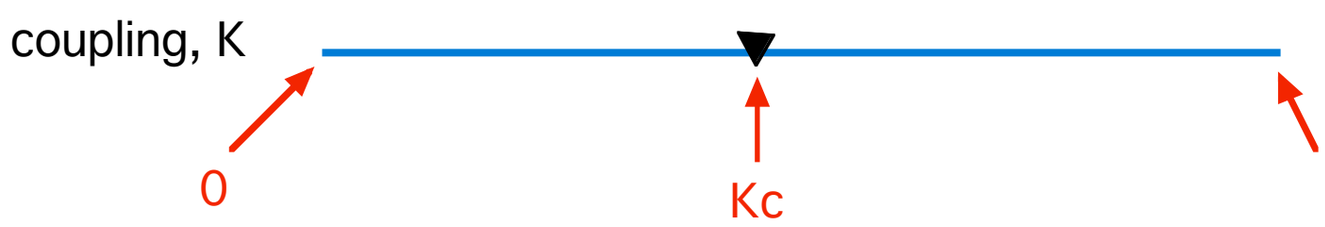
One can do many more roughly analogous calculations and compare with experiment and numerical simulation. Everything works!

However notice that this is not a complete theory. We have no way to find x from theory.

renormalizations of couplings



- stable fixed point
- unstable fixed point



Homework:

Add a term in $\Sigma_j (h\ \sigma_j)$ to the weighting function, W, for the one dimensional Ising Hamiltonian. Find the value of the average spin in the presence of a small magnetic field h. Define the magnetic susceptibility as the derivative of the magnetization with respect to h at fixed K. Show that this susceptibility diverges as K goes to infinity. Shows that it is proportional to a sum of fluctuations in the magnetization.

The three-state Potts model is just like the Ising model except that its "spin" variable σ_j can take on three values $= -1, 0, 1$. It has $w(\sigma_j, \sigma_{j+1}) = K$ if the two variables are the same and zero otherwise. Find the partition function and coherence length of the one dimensional model. How does the renormalization work for ?

What is the critical temperature of the three-state Potts model on the square lattice in two dimensions?