

The Trouble with Cycles in the N - K model

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Abstract

In studying dynamical systems models, like Kauffman's N - K model, one very often analyzes the behavior by using the structure of the cycles in the system. One gives distributions of cycle-lengths, basin sizes, and of transients. Here we argue that this kind of characterization is very far from robust, and is hence perhaps not very useful for describing physical or biological systems. We propose an alternative based on the analysis of the effects of small amounts of noise upon such systems. For the original papers using this kind of noise see [1, 2].

This note is drawn from a review paper [3] and from the research of Xiaohui Qu et al. [4]. It is part of a robustness study at the Santa Fe Institute supported by the Packard Foundation.

1 Structure of Models

A very general dynamical systems model starts from N Boolean elements or variables $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$. It has a time stepping, in which each of these elements is given by a function of the other elements. More precisely, the value of σ_i at time $t + 1$ is determined by the value of its K_i *controlling elements* $\sigma_{j_1(i)}, \sigma_{j_2(i)}, \dots, \sigma_{j_{K_i}(i)}$ at time t . In symbols,

$$\sigma_i(t + 1) = f_i(\sigma_{j_1(i)}(t), \sigma_{j_2(i)}(t), \dots, \sigma_{j_{K_i}(i)}(t)) \quad (1)$$

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where f_i is a function associated to the i -th element that depends on K_i arguments. The functions are independent of time. For convenience we will denote by Σ_t the state of the system at time t , namely,

$$\Sigma_t = \{\sigma_1(t), \sigma_2(t), \dots, \sigma_N(t)\}$$

If each variable can take on a finite number of values, the system will eventually fall into cycles. Much work has been done on characterizing the behavior of the cycles in general models of this kind.

2 Non-robustness of Cycles

A cycle forms when the system returns to a configuration which is exactly the same as one it has previously visited. A demand for an exact return should be viewed not as a single constraint but in fact as N constraints upon the configuration. Such a strict demand makes the properties of cycles quite special and delicate. For this reason, *a study of cycles is probably not what one would want for understanding the possible physical or biological consequences of the models like the N - K model.* The statement just made flies in the face of a very large body of work, some of which we have just described. We should, for this reason, argue for this statement with some care.

Why don't we believe in cycles?

1. The characterizers of cycles are neither intensive nor extensive variables.
2. With exponentially short cycles in localized regions one gets power laws overall. (See section 4 below.).
3. The critical situation has very many, short cycles which are not observed when one starts from randomly picked starting points.
4. Attractor basins are complex in character, being at best multi-fractal for large chaotic systems.
5. In a large system you must wait so long to see a cycle that it cannot be really important.

6. In a large chaotic system, changing one rule changes the cycles quite a bit.
7. In both glasses and biological systems one wants to characterize the system by things which are very robust.

We do believe that generic properties of cycles are important though, in that they characterize general aspects of dynamical systems. Nonetheless, the huge fluctuations throughout realizations in such important quantities as cycle lengths and number of different attractors, calls for other types of characterizations. Real networks, whether they are genetic or neural or of any other kind, are always subjected to external perturbations. The robustness in the dynamics of the network can not rely on quantities which change dramatically with perturbations. Hence, it is important to characterize the dynamical properties of the network in the presence of noise, trying to find out which kind of quantities are preserved under the influence of noise, and which ones are not.

3 Characterization via noise

The addition of noise to a map provides a possibility for generating additional information about the behavior of the models. Noise naturally blurs out the sharpness of behavior, making for a more “fuzzy” characterization. Noise is, then, a natural way to get away from the difficulty posed by the overly-precise characterization provided by the cycles.

Unfortunately, most previous work on the N - K model does not include noise. We do wish to point to two papers [1, 2] in which noise has been used to probe N - K behavior. For reasons which will become more evident later on, we describe these papers respectively as a “crossing paper” and a “convergence paper”. These papers both break the precision of the dynamical rule of equation (1) by saying that the rule is broken with a probability r

$$\sigma_i(t+1) = \begin{cases} f_i(\sigma_{j_1(i)}(t), \dots, \sigma_{j_{K_i}(i)}(t)) & \text{with probability } 1-r \\ \neg f_i(\sigma_{j_1(i)}(t), \dots, \sigma_{j_{K_i}(i)}(t)) & \text{with probability } r, \end{cases} \quad (2)$$

which can also be written in the alternative form:

$$\sigma_i(t+1) = \begin{cases} f_i(\sigma_{j_1(i)}(t), \dots, \sigma_{j_{K_i(i)}}(t)) & \text{with probability } 1 - 2r \\ 1 & \text{with probability } r \\ 0 & \text{with probability } r. \end{cases} \quad (3)$$

This equation can be described as providing probabilities r for the two possible values of the outcome, independently of the value of f_i . In [1] the rules are described in terms of a temperature T , related to r by

$$r = \frac{1 - \tanh(1/T)}{2}. \quad (4)$$

Both groups examine the development of two or more different initial configurations using the same realizations. They also apply exactly the same rules and the same probabilistic choices to the different configurations. So far, both papers are essentially similar. There are two kinds of differences, the first being the choice of measurement, and the second being the way they apply equations (2) and (3).

The convergence paper starts with two or more randomly chosen initial configurations, $\Sigma_0^1, \Sigma_0^2, \dots, \Sigma_0^m$, and calculates the resulting trajectories step by step:

$$\begin{array}{l} \Sigma_0^1 \rightarrow \Sigma_1^1 \rightarrow \Sigma_2^1 \rightarrow \dots \Sigma_\tau^1 \\ \Sigma_0^2 \rightarrow \Sigma_1^2 \rightarrow \Sigma_2^2 \rightarrow \dots \Sigma_\tau^2 \\ \dots \\ \Sigma_0^m \rightarrow \Sigma_1^m \rightarrow \Sigma_2^m \rightarrow \dots \Sigma_\tau^m \end{array}$$

At each step, and for each i , a choice is made among the three branches of equation (3), and that choice is equally applied to the m configurations.¹ The calculation continues until two of the m configurations become identical (say for example $\Sigma_\tau^1 = \Sigma_\tau^2$). The time needed to achieve the convergence is noted. We will denote this time by τ_m , stressing the fact that m configurations are being analyzed.

In some ways, the convergence calculation is more complicated than the crossing one. The noise as defined by equation (3) tends to produce convergence because it is applied equally to all trajectories and because makes the

¹The two-branch versus three-branch methods become inequivalent when they are applied to several configurations at once.

values of the elements to be equal in all trajectories. In the limit of infinite N , the system shows three phases: the low noise phase in which almost always trajectories will not converge, the high noise phase in which trajectories will always converge, and a separating critical phase. These are respectively described as low temperature, high temperature, and critical phases.

In reference [1], the authors examine $K = 4$ and large- N , picking the noise-levels $r = 0.15$ (low temperature phase), $r = 0.25$ (critical phase), and $r = 0.35$ (high temperature phase). Without noise, the N - K model would show chaotic behavior, and very long cycles. In the presence of noise the average of $\ln \tau$, with τ being the convergence time, diverges as N goes to infinity. The behavior of the convergence time for two orbits, τ_2 , can be calculated in the annealed sequential update case, and the results compared with simulations for both annealed and quenched system. In both simulation and theory, the three phases are characterized by having different forms of divergence of $\langle \ln \tau_2 \rangle$ with N (see Fig. 1). At low temperatures the divergence is linear: $\langle \ln \tau_2 \rangle \sim N$. In the critical phase $\langle \ln \tau_2 \rangle$ diverges linearly with $\ln N$ as $\langle \ln \tau_2 \rangle = 0.5 \ln N$. The weakest divergence occurs in the high temperature phase in which $\langle \ln \tau_2 \rangle$ varies as $\ln \ln N$. These N -dependencies describe the variation of the number of elements forming the barrier to having two configurations merge into one another. The noise causes these elements to be identical and then the merge occurs. Thus, for example, in the low temperature phase, we must bring to equality a finite fraction of all the elements in the system in order to have the convergence.

In some loose sense, these numbers measure the size of the barriers which hold together the attractors for this system. But it is hard to know what the attractors themselves might be. Because r is relatively large, these presumed attractors are probably not the cycles of the original system. In fact, the behavior of τ_2 is much the same for the annealed system (which has no cycles) as for the quenched system, which does have cycles at $r = 0$. Nobody has yet investigated the limiting case as r goes to zero. It might be most interesting to look at this limit, particularly in association with a limit which keeps the system critical (say K goes to 2).

Let us consider the case in which more than two trajectories are analyzed. Start with m trajectories and let τ_m measure the first time when any two of these have converged. If there are only M large or important basins, one

might well expect $\langle \ln \tau_m \rangle \ll \langle \ln \tau_M \rangle$ for $m > M$. Instead one observes that

$$\langle \ln \tau_m \rangle = \langle \ln \tau_2 \rangle - \ln \frac{m(m-1)}{2},$$

in both theory (in the annealed approximation) and simulation. This form indicates an indefinitely large number of attractors, all with basins of comparable size.

The crossing paper considers two different configurations Σ_0^1 and Σ_0^2 , which can belong either to different attraction basins or to the same basin of attraction, and then iterates forward, noting all configurations $\{\Sigma_0^1, \Sigma_1^1, \Sigma_2^1, \dots\}$ and $\{\Sigma_0^2, \Sigma_1^2, \Sigma_2^2, \dots\}$ they produce. In each step of iteration, and for each i -value, a choice is made between the two branches of equation (2), and that choice is applied to both configurations. This continues until the time τ in which one of the two trajectories attains a configuration previously entered by the other one (for example, if Σ_τ^2 is equal to one of the configurations $\{\Sigma_0^1, \Sigma_1^1, \dots, \Sigma_\tau^1\}$). The calculation is terminated at this crossing event. The measured quantity is the “time” τ needed to produce the crossing.

In the absence of noise, if two initial configurations belong to different basins of attraction, the time for the two subsequent trajectories to cross is infinite. In the presence of noise, there is a chance for each trajectory to “jump out” of its basin of attraction, exploring a bigger part of the state space. The two trajectories will have a number of opportunities equal to τ^2 , to cross one another before τ steps have elapsed. If the size of the space being explored by the trajectories is $\Omega(r)$, then the typical time for the crossing will be

$$\tau \approx \Omega(r)^{1/2}. \quad (5)$$

Miranda and Parga simulated the system and measured τ as a function of r and N for the critical Kauffman net in which $K = 2$. Their result for large N may be summarized as

$$\Omega(r) = \Omega^{A(r)}, \quad (6)$$

where $\Omega = 2^N$ is the volume of the state space in the system.

In the work by Miranda and Parga the “fractal” exponent $A(r)$ was not estimated accurately. This work was recently extended by X. Qu et al., who consider larger values of noise and different connectivities of the network ([4]). The authors analyze two cases to compute the crossing time, when the two initial configurations belong to the same basin of attraction, and when

they belong to different basins. We will denote these two crossing times by τ_s and τ_d respectively. Fig. 2a shows the average crossing times τ_d and τ_s as functions of r , for a net with $N = 20$ and $K = 2$. As can be seen, when r is close to its maximum value 0.5, both times are practically the same. In fact, Qu et al. have shown that for large values of r both τ_s and τ_d behave as

$$\tau_{d,s} \approx \frac{\sqrt{\pi}}{2} 2^{N/2} \left\{ 1 + \frac{(1-2r)^2}{2^K} \right\}^{-N/2}. \quad (7)$$

The above expression agrees with equations (5) and (6) by identifying

$$A(r) = 1 - \frac{\ln \left[1 + (1-2r)^2/2^K \right]}{\ln 2}. \quad (8)$$

In contrast, when r is close to 0 the behavior of τ_d and τ_s differ substantially. For $r \rightarrow 0$ the divergence of τ_d is given simply by

$$\tau_d \approx C_1/r + \tau_0 \quad (9)$$

where C_1 only depends on K and N , and τ_0 is the value of τ_s at $r = 0$.

The complete analytical expression of $A(r)$, valid in the whole interval $[0, 1/2]$ is not known yet. Fig. 2b shows a plot of $A(r)$ obtained by numerical simulations. It is interesting to note that the attractor has a fractal volume which depends upon r . Once again, one is frustrated because one does not know what the attractor might be. It is once again probably not anything directly related to a cycle, since starting points in the same cycle or in different cycles both give the same τ -values for the higher values of N and r . Here too one might guess that studies with smaller values of r might shed light on the N - K model attractors.

4 Independent Subsystems

Here we do a calculation which is useful in interpreting results involving cycles. Imagine a system composed of N independent subsystems. Each subsystem has a probability ρ_l of having a cycle of length l . We imagine that ρ_l gets quite small for large l and ask what is the chance of finding a long

cycle in the entire system. Notice that the chance of not finding a piece with a cycle of length l in the entire system is

$$q_l = (1 - \rho_l)^N \approx \exp(-N\rho_l).$$

If then, ρ_l varies exponentially with l , namely, if

$$\rho_l = A \exp(-\alpha l), \tag{10}$$

then we might expect to find parts with all cycle lengths up to

$$l_{mx} = (\ln N)/\alpha \tag{11}$$

(so that q_l is not that small, say of order $q_l \sim e^{-A}$). To make a long cycle in the entire system, one puts together many sub-cycles of different lengths, $l_i, i = 1, 2, \dots, N$. The total cycle length, L , is the smallest number divisible by each of the l_i 's. Then L will be a product of all prime numbers, p_r , which are less than l_{mx} , each raised to a power s_r which is the largest integer for which the inequality

$$\left[\frac{l_{mx}}{p_r^{s_r}} \right] \geq 1$$

is satisfied ($[x]$ being the integer part of x). Hence, to a decent approximation, the largest cycle length L_{mx} will be

$$L_{mx} \approx (l_{mx})^{\pi(l_{mx})},$$

where $\pi(l)$ is the number of primes less than l , which can be estimated in the asymptotic limit of large l as ²

$$\pi(l) = l/\ln l.$$

In the end then, the longest cycle length L_{mx} obeys

$$\ln L_{mx} \approx \pi(l_{mx})(\ln l_{mx}) \approx l_{mx},$$

²A better approximation is $\pi(l) = 1/(\ln l - 1)$. It is also valid that

$$\frac{l}{\ln l} < \pi(l) < 1.0423 \frac{l}{\ln l}$$

for all $l > 10$.

so that

$$L_{mx} \approx N^{1/\alpha}. \quad (12)$$

We have reached the remarkable conclusion that even though the probability of long cycles in each component of the system falls exponentially, the typical maximum cycle length in the entire system depends algebraically upon the size of the system. This calculation does not apply directly to Kauffman nets because we have not accounted for the fact that the different modules have different distributions ρ_l , but nonetheless it is instructive.

References

- [1] Golinelli, O. and Derrida, B. *Barrier Heights in the Kauffman Model*. Journal De Physique, **50** (1989) 1587-1601.
- [2] Miranda, E. N. and Parga, N. *Noise Effects in the Kauffman Model*. Europhysics Letter, 10(1989) 293-298.
- [3] Aldana, M., Coppersmith, S., Kadanoff, L. *Boolean Dynamics with Random Couplings*. In press.
- [4] Qu, X., Kadanoff, P., Aldana, M. *Numerical and Theoretical Studies of Noise Effects in the Kauffman Model*. To be submitted.

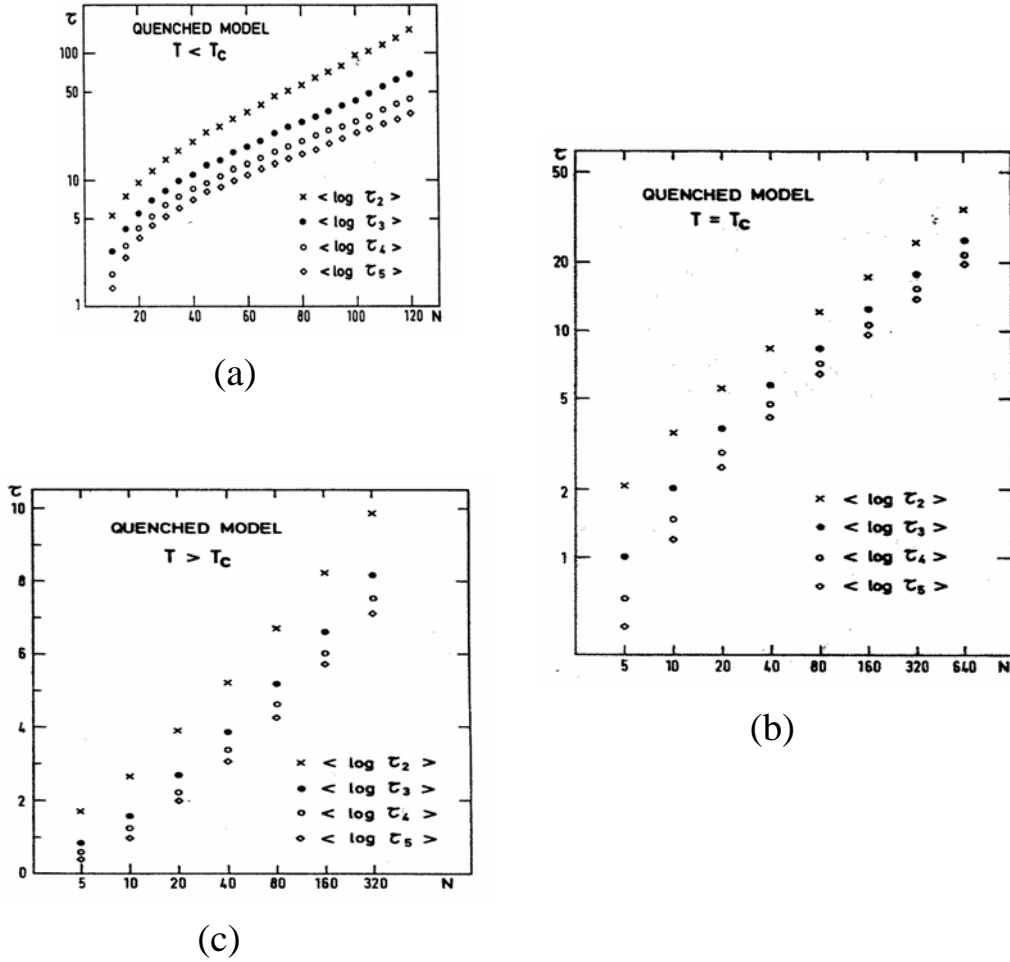


Figure 1: Plots of $\langle \ln \tau_m \rangle$ versus N , taken from reference [1]. In all the cases the connectivity of the network was $K = 4$. The three graphs correspond to three values of r : (a) Low temperature phase, $r = 0.15$; (b) Critical phase, $r = 0.25$; (c) High temperature phase, $r = 0.35$.

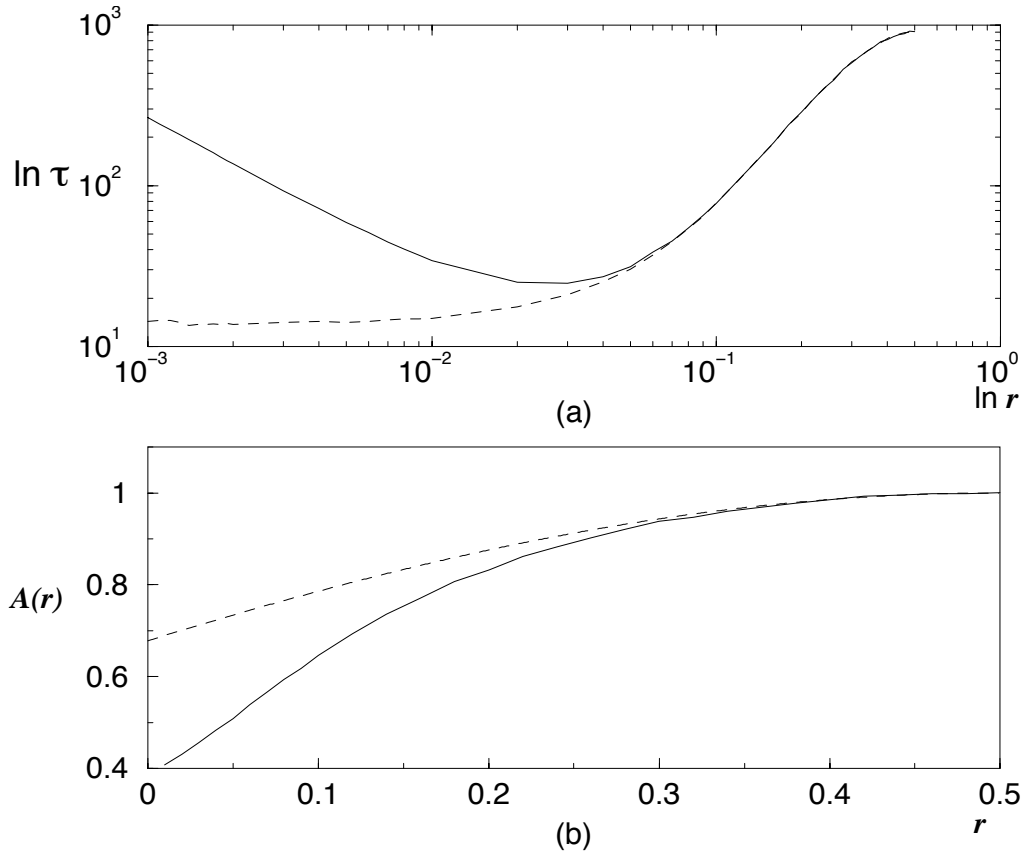


Figure 2: (a) Log-log plot of the crossing time τ as a function of the level of noise r . The solid curve is τ_d , the crossing time starting from different attractors, whereas the dashed curve corresponds to τ_s , the crossing time starting from the same attractor. (b) The index $A(r)$ describing the fractal dimension of the attractor plotted as a function of r . The solid curve is the result of the numerical simulation, obtained as $A(r) = 2 \ln \tau / \ln \Omega$, and the dashed line is the theoretical prediction given in (8). Note that these curves become identical when $r \rightarrow 0.5$. Both graphs (a) and (b), calculated for a situation with $K = 2$ and $N = 20$, were taken from reference [4].