Crumpling a Thin Sheet

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Crumpled sheets have a surprisingly large resistance to further compression. We have studied the crumpling of thin sheets of Mylar under different loading conditions. When placed under a fixed compressive force, the size of a crumpled material decreases logarithmically in time for periods up to three weeks. We also find hysteretic behavior when measuring the compression as a function of applied force. By using a pretreating protocol, we control this hysteresis and find reproducible scaling behavior for the size of the crumpled material as a function of the applied force.

If you, the reader, were to rip a page from this journal and crumple it, squeezing it with your hands into a ball as tightly as you can, the resulting object is still more than 75% air. What gives this crumpled sheet its surprising strength and how does the ultimate size of the sheet depend on the forces applied?

The energy stored in crumpled sheets has been investigated in the limit where the thickness, δ, of the sheet is much less than the lateral dimension, L (e.g., the diameter of a circular sheet) [1–5]. Analysis of these objects has focused on two characteristic structures caused by the crumpling: singular, conical points and the curved ridges, which store most of the energy, connecting them. As the sheet is crumpled farther, the ridges must collapse so as to remain within the confining container, and an increasing number of smaller ridges must form. In the limit where the diameter of the container, D, is much smaller than the initial size of the sheet but still much larger than the sheet thickness ($\delta \ll D \ll L$) a scaling relation is predicted between the energy stored and the size of the resulting crumple: $E \propto D^{\beta}$. This prediction assumes that the forces resisting crumpling are conservative. One knows, however, that frictional forces are involved as the sheet rubs against itself and against the constraining walls. How do these forces affect the resulting energy balance? In addition, plastic flow can occur in regions of high curvature effectively constraining the geometry of the energy bearing structures.

The notion of scaling properties describing a crushed sheet goes back to the initial experiments of Kantor *et al.* [5] Further investigations of such sheets were made by Gomes and collaborators [6,7]. An analytical theory and experiments have focused on the conical singularities and incipient ridges [8,9]. More broadly, scaling behavior of force with compression in tenuous structures was reported for random fibrous material by Baudequin *et al.* [10] and for colloidal aggregates by several groups [11]. Here we report distinctive forms of scaling not seen in those previous studies.

To study how the external dimensions of a crumpled sheet depend on the confining force, we have placed a large circular ($L = 34$ cm) sheet of thin ($\delta = 12.5$ μm) aluminized Mylar [12] under a weighted piston in a plastic cylindrical cell of diameter $D = 10.2$ cm. Sheets of different size were also used with no obvious change in the results. The cylinder is surrounded by a box to protect it from air currents in the room and is mounted on an optical table to minimize the effects of vibration. We measure compression by measuring the height, h, of the piston above the base. No special care is taken in the initial crumpling that allows the large sheet to be placed within the cylinder.

After a mass, M, is placed on the piston, the piston settles to a new height held up by the crumpled sheet below it. However, this behavior is not as simple as one might have expected: h continues to decrease long after the mass is initially introduced. As shown in Fig. 1(a), over seven decades in time, 2×10^{-1} sec $< t < 2 \times 10^{6}$ sec, the entire duration of the experiment, the Mylar height decreases logarithmically:

$$h = a - b \log(t/\mathrm{sec}), \qquad (1)$$

where a and b are constants. Even after three weeks, the piston did not reach its asymptotic height. Similar relaxation occurred in other tenuous materials, as shown in the insets of Fig. 1(a). Absorbent tissue paper [13] showed a $\log t$ dependence similar to that of the Mylar. Cotton balls [14] showed a slightly smaller relaxation over several weeks, in which the logarithmic dependence extended only up to 10^{4} sec. Although this logarithmic behavior is robust and is found each time the Mylar is compressed, the constants a and b fluctuate between runs with no monotonic evolution of those constants as the sheet is crumpled more times. Figure 1(b) shows a weak correlation between a and b for the same 200 g load on the piston.

Albuquerque and Gomes [7] studied stress relaxation of crumpled aluminum foil under fixed strain and found stretched-exponential behavior quite different from the logarithmic dependence of strain that we measured at fixed stress. Aluminum foil is much more maleable than the Mylar we have used and presumably has more
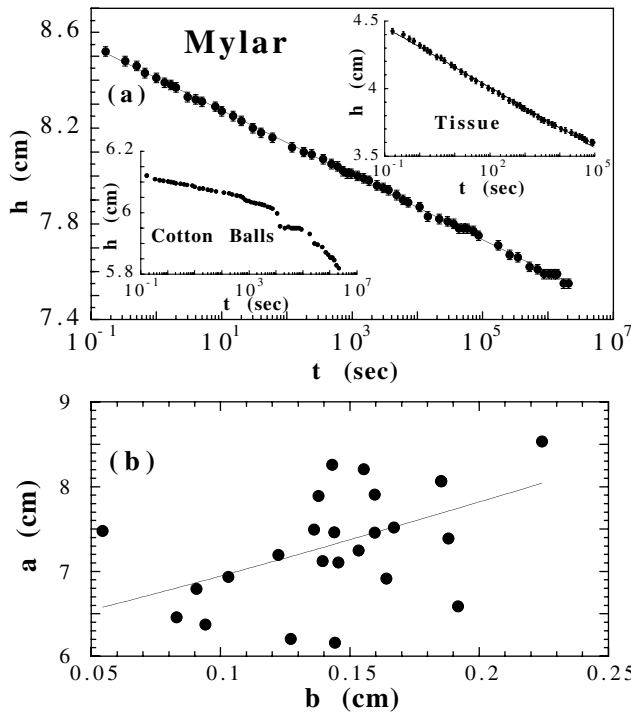
FIG. 1. (a) Time dependence of the height of a Mylar sheet compressed by a mass $M = 200$ g. The straight line is a fit to Eq. (1) with $a = 8.4$ cm and $b = 0.14$ cm. Insets show the time dependence of the height for porous tissue and for cotton balls under the same conditions: $M = 200$ g. (b) The constants a and b from Eq. (1) for a series of runs under nominally identical conditions. No obvious evolution of the values was found as the sheet was crumpled successively more times.
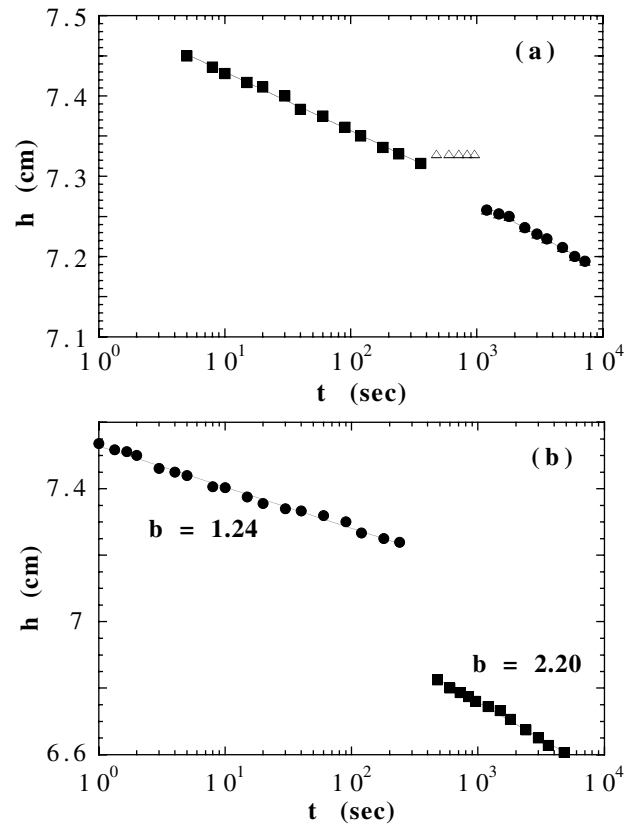

FIG. 2. (a) Time dependence of the height of a compressed Mylar sheet before and after the compression was stopped. The initial time dependence of the height of a Mylar sheet after a 200 g mass has been placed on it is shown by the filled squares. After approximately 500 sec, the piston is lifted and fixed in place. After 1000 sec, the piston is released and the subsequent relaxation is shown by the filled circles. (b) $h(t)$ for a compressed Mylar sheet before and after external vibrations are introduced. Initially the container was at rest. After approximately 200 sec, lateral vibrations with an acceleration of 2.5 m/sec² at 7.5 Hz were begun. The resulting logarithmic slope, b, nearly doubled.

plastic flow along the ridges and vertices. This may be the origin of the different relaxation kinetics. Menon [15] has found similar log$t$ behavior as our results in stress relaxation.

Relaxation continues even if the compression is stopped. Figure 2(a) shows the time dependence of the height of a Mylar sheet after a 200 g mass has been placed on it. After 500 sec. the piston is lifted very slightly and fixed in place so that it can no longer compress the material. After 1000 sec, the piston is released and quickly sinks to a new position comparable to where it would have been had it never been restrained; thereafter it relaxes at nearly the same logarithmic rate as it did initially.

What is responsible for the slow decrease in height over these extended periods? External vibrations affect the compression. Figure 2(b) shows $h(t)$ both before and after the container is vibrated *laterally* with an acceleration of 2.5 m/sec² at 7.5 Hz. Initially, the container was at rest and the logarithmic slope, b, was determined. After approximately 200 sec, vibrations were begun and b nearly doubled. This change, while substantial, is not so large as to indicate that the slope in the quiescent state is caused by the ambient vibrations in the room which are less than 10^{-2} m/sec².

This logarithmic decay of h suggests activated relaxation (with an effective temperature T_{eff} introduced to account for external vibrations [16]) of the sheet between metastable minima separated by energy barriers, ΔE: $dh/dt = -\zeta \exp[-\Delta E/(k_B T_{\text{eff}})]$. As h decreases, the crumpled sheet becomes progressively stiffer suggesting that the barriers are increasing. To lowest order (i.e., $\Delta E \approx E_0 - Ch$), we find

$$dh/dt = -\zeta \exp[-(E_0 - Ch)/(k_B T_{\text{eff}})], \quad (2)$$

which has a solution of the form of Eq. (1), where $a = d_1 + E_0/C - (k_B T_{\text{eff}}/C)\ln(C\zeta/k_B T_{\text{eff}})$ and $b = k_B T_{\text{eff}}/C$, where d_1 is a constant. As T_{eff} increases (e.g., due to external vibrations), the logarithmic slope, b, also increases as observed in the experiments. If the precise initial crumpling determines E_0 and C, then a and b should be correlated between runs: $a = d_1 + b(\ln b + d_2)$, where $d_2 = E_0/k_B T_{\text{eff}} - \ln\zeta$. The curve in Fig. 1(b) shows

the fit with $d_1 = 6.2$ and $d_2 = 9.7$. We note, however, that d_1 and d_2 may also vary between trials.

Thus far, we have reported the time dependence of the height, h, under a fixed weight. We are also interested in determining the effect of the piston mass, M, on the height. There are two problems which interfere with obtaining such data in a straightforward manner. First, as we have emphasized, h varies logarithmically with time and it is thus impractical to find the final height of a crumpled sheet under a given load. Second, there is large hysteresis so that different loading procedures produce different values of h. Such hysteresis is already suggested by the variation between runs of the constants, a and b, describing the relaxation. Despite these problems, we can obtain reproducible results by following two protocols for handling the crumpled sheets.

In order to avoid the problems created by the slow relaxation, we measure the height after a fixed interval of time (e.g., 100 sec) and use this value, instead of the unattainable asymptotic ($t = \infty$) value, for comparison between different loadings. Such a procedure means that we neglect relaxation processes slower than 100 sec. To obtain meaningful results, we confine ourselves to comparing the piston height for relatively large changes in the mass, ΔM, so that $\Delta h_{\Delta M}$ (the change in height measured at $t = 100$ sec due to the increased piston mass) is much greater than $\Delta h_{\Delta t}$ (the change in height due to waiting extra time after the mass has been added, e.g., between 100 sec and 1 day): $\Delta h_{\Delta M} \gg \Delta h_{\Delta t}$. In what follows, we measure $h_{100}(M)$, the height at $t = 100$ sec after a mass, M, has been added to the piston.

Dealing with the second problem of hysteresis requires a protocol for preparing a sheet prior to measurement. Hysteresis is evident if we start with a small initial mass and measure (as prescribed above) the height after subsequent increments ΔM have been added until a large final mass is applied. If we then remove this load and repeat the measurement with the initial mass, we find that the newly measured height is smaller than that initially measured for the same mass.

In order to avoid this hysteresis, we have followed the protocol of Baudequin *et al.* for studying compressed glass wool [10]. We pretrain a sheet by first crumpling it under a mass M_1 for a period of one day. The mass is then removed and we measure $h_{100}(M)$ for $M < M_1$ [as shown for $M_1 = 2.6$ kg by the open circles in Fig. 3(a)]. This curve is reproducible (shown by the ×'s) as long as M_1 is not exceeded. However, if we exceed M_1 and place a larger mass on the piston, we no longer obtain the same curve. With this larger mass $M_2 = 4.5$ kg, we obtain a new curve (shown by the solid triangles) that is itself reproducible as long as M_2 is not exceeded. We continue this measurement to obtain a family of curves each of which corresponds to a pretraining mass M_i. For each curve, considered separately, we find a scaling relation:
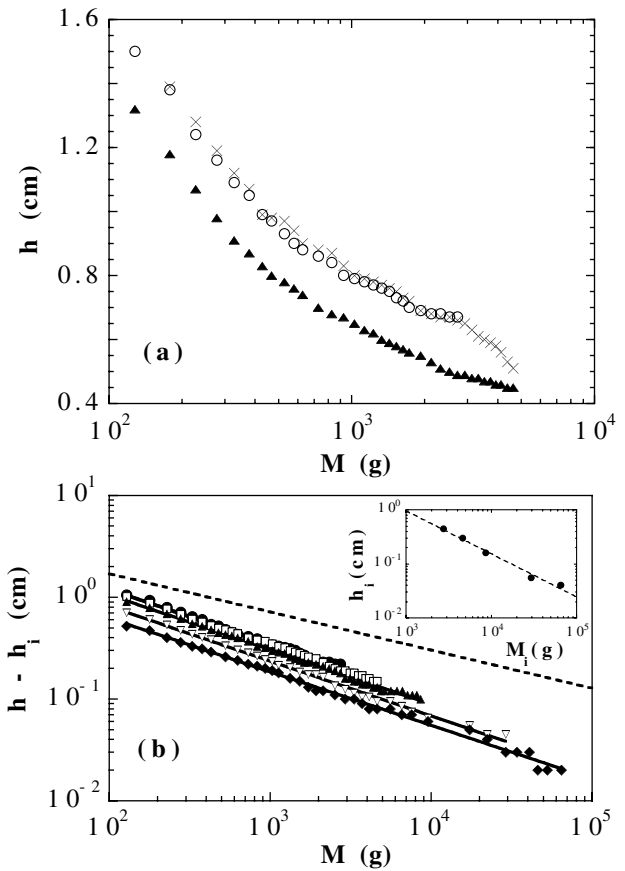

FIG. 3. (a) Hysteresis in the height versus mass. A crumpled Mylar sheet is first placed under a mass $M_1 = 2.6$ kg for one day. The mass is then removed and $h_{100}(M)$ is measured for $M < M_1$ (open circles). If M_1 is not exceeded, this curve is reproducible (×'s). If M_1 is exceeded, a different curve $h_{100}(M)$ is found (solid triangles). (b) Log[$h_{100}(M, M_i) - h_i$] versus log$M$ for five separate pretreating masses M_i. Solid lines are fits to Eq. (3) with $\alpha = 0.53$. The dashed line shows the predicted $h(t)$. The inset shows scaling between the asymptotic height, h_i, and the pretreating mass, M_i.

$$h_{100}(M, M_i) = h_i + cM^{-\alpha}, \qquad (3)$$

where h_i is the asymptotic value of the height for the pretreating mass M_i. Such a family of curves is shown in Fig. 3(b). We find $\alpha = 0.53 \pm 0.04$ for all pretreating masses. This is smaller than the value $\alpha = 0.67$ found by Baudequin *et al.* [10] for fibrous glass wool. The inset of Fig. 3(b) shows a scaling relation between the asymptotic height, h_i, and M_i: $h_i \propto M_i^{-\gamma}$ with $\gamma \approx 0.8$. Despite the problems associated with the logarithmic relaxation and hysteresis, Fig. 3(b) shows that it is possible, under well-defined conditions, to obtain reproducible data on crumpled sheets.

We can predict the scaling relation between M and h if we assume a small volume fraction and a uniform network of ridges between nearly flat facets of characteristic size X that depends on compression. The energy of each facet is proportional to $\kappa(X/\delta)^{1/3}$, where κ is the bending modulus of the sheet and δ is its thickness [1–5].

Facets are oriented at random, so there is roughly one facet per volume X^3. Thus the energy contained in the volume V is $E \propto V/X^3 \kappa (X/\delta)^{1/3}$. The space occupied by one facet is large enough to contain a stack of X/δ facets so that $\phi \equiv \delta/X$ is a volume fraction. Macroscopically, $\phi = [\pi \delta (L/2)^2]/[\pi h (D/2)^2]$, where h is the height of the container, D is its diameter, and L is the diameter of the sheet. E can be reexpressed:

$$E = \kappa V/\delta^3 (X/\delta)^{-8/3}$$
$$= (\kappa/\delta^3)[\pi h (D/2)^2]\phi^{8/3} \propto h^{-5/3}. \qquad (4)$$

On the other hand, the energy E supplied by the applied force Mg is given by $E = \int_\infty^h M(h)g\,dh \approx Mgh$. As noted previously, we have assumed here that all the work goes into ridge energy. We conclude that $Mgh \approx h^{-5/3}$, so that $h \propto M^{-3/8}$. This predicted value of $\alpha = 0.375$ is to be compared with the measured $\alpha = 0.54$. A more careful estimate [17] yields $h \approx 1.7$ cm when $M = 0.1$ kg. The predicted height, shown by the dashed line of Fig. 3(b), is a factor 2 higher than those measured for the smallest masses.

The computer simulations of Lobkovsky *et al.* [1] indicated that the asymptotic scaling worked well when the volume fraction was no more that a few tenths of a percent. However, the experiments are well outside this regime, where the volume fraction is large (over one-half). The size of the observed facets is far from uniform, and this nonuniformity could alter the stored energy. Further, we neglect friction which is expected to strengthen the crumpled sheet by a factor of order unity. Plastic deformation is expected to weaken it by at most a factor of order unity. Considering these differences between the experiment and the theoretical model, the agreement between the two exponents is surprisingly good.

The data presented here suggests that the answer to our question in the first paragraph, about how the ultimate size of a sheet depends on the forces applied, has a complicated answer. The $\log t$ dependence found in the relaxation suggests that there is dissipation, either due to friction or to plastic flow in regions of large curvature. It is unclear to what extent an analysis based only on conservative forces, which does not consider such effects, can fully describe the energetics of crumpling in real materials. Despite that caveat, our data nevertheless do support a scaling picture of the crumpling process: After a sheet has been appropriately pretreated, the height approaches an asymptotic value as a power law.

The results suggest several extensions such as studying how the exponent α depends on size and thickness in different materials. The most pressing question is the origin of the logarithmic relaxation. We do not know if the behavior occurs because of plastic flow in the material or because of frictional sliding at the contacts between sheets. Neither effect is included in the theoretic or simulation studies of crumpling. In order to determine the role of plastic flow in regions of large curvature, one could measure the time-dependent relaxation in rubber sheets where such plasticity would be minimized.

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