Physics 251 Quiz 1, January 19, 2006 Name .....

The function $f(x) \equiv r \sin x$ for $0 < x < \pi$ is similar in form to the Logistic map for $0 < x < 1$. We consider the sequence of values x_0, $x_1 = f(x_0)$, $x_2 = f(x_1)$, etc.

1) For the parameter r less than some r_1, this sequence of x_n converges to zero. Find r_1. You may assume that the starting x_0 is small.
2) For some value of r denoted r_2, the sequence of x_n's converges to a superstable fixed point. What is the value of r_2 and the converged value x^*. Again, feel free to assume that x_0 is near x^*.
3) For some value of r denoted r_3 there is a fixed point at $x^* = 2$. Is this fixed point stable or unstable? Explain your answer. The following trigonometric identities might be useful:

$$\cos^2 x + \sin^2 x = 1; \quad \sin 2 = 0.909; \quad \cos 2 = -0.416; \quad \tan 2 = -2.185$$

Solution:

1) $x_1 = r \sin x_0 \simeq r x_0$. For all small x_0, $x_1 < x_0$ iff $r < 1$. Therefore $x_n \to 0$ when $\boxed{r < 1}$. Note that we are concerned with the sequence $x_n = f(x_{n-1})$, not the sequence of Newton-Raphson estimates.
2 A superstable fixed point x^* is one where $df/dx|_{x^*} \equiv f'(x^*) = 0$. At this point $0 = r \cos(x^*)$, so $\cos x^*$ is zero. thus $\boxed{x^* = \pi/2}$. The corresponding r_2 is given by the fixed-point condition $r_2 \sin x^* = x^*$, so $r_2 \sin \pi/2 = \pi/2$, or $\boxed{r_2 = \pi/2}$.
3) The fixed-point condition says $r_3 \sin 2 = 2$, so $r_3 = 2/\sin 2$. The stability condition says

$$1 > \left| \frac{df}{dx} \right| = |r_3 \cos 2| = |2 \cos 2 / \sin 2| = |2/(-2.185)|$$

Evidently the stability condition is satisfied, so the fixed point is $\boxed{\text{stable}}$.