

**1. A two-dimensional fixed point**

A certain periodically-driven oscillator with period  $\tau$  (not DoubleWell) has has position co-ordinate  $x$  and momentum co-ordiante  $p$ . If the oscillator is at any given  $(x, p)$  at time 0 then at a time  $\tau$  the new  $(x_1, p_1)$  given by

$$x_1 = x + p; \quad p_1 = \frac{3}{4}x + x^2 + p^2$$

Evidently the point  $x = p = 0$  is a fixed point.

- a) Is this a stable or an unstable fixed point, or is this question ambiguous? Explain your reasoning.
- b) Find two points  $(x_a, p_a)$  and  $(x_b, p_b)$  very close to the origin such that the  $(x, p)$  position after one cycle moves directly towards or away from the fixed point. That is,  $(x_a, p_a)$  maps into a point  $(x_{a1}, p_{a1}) = \lambda_a(x_a, p_a)$ , for some constant  $\lambda_a$ , and similarly for  $(x_b, p_b)$ . It is sufficient to determine the ratios  $x_a/p_a$ , and  $x_b/p_b$ .

**Solution:**

- a) The stability of a fixed point tells the result of displacing the initial point slightly from the fixed point. In two dimensions this stability depends on the direction of displacement. One analyzes this by finding directions that move directly towards or away from the fixed point, so that the displacement  $\delta x, \delta p$  satisfies

$$\begin{pmatrix} \delta x_1 \\ \delta p_1 \end{pmatrix} = \lambda \begin{pmatrix} \delta x \\ \delta p \end{pmatrix}$$

Since the displacements are arbitrarily small, the  $x_1$  and  $p_1$  are linear functions of  $\delta x, \delta p$ , so that

$$\begin{bmatrix} \partial x_1 / \partial x & \partial x_1 / \partial p \\ \partial p_1 / \partial x & \partial p_1 / \partial p \end{bmatrix} \begin{pmatrix} \delta x \\ \delta p \end{pmatrix} = \lambda \begin{pmatrix} \delta x \\ \delta p \end{pmatrix}$$

These derivitives was called  $g_x, g_p, h_x, h_p$  in the notes. The derivitives are supposed to be evaluated at the fixed point:  $x = p = 0$ . The  $x^2$  and  $p^2$  terms in  $p_1$  are negligible. Thus the matrix is

$$\begin{bmatrix} 1 & 1 \\ 3/4 & 0 \end{bmatrix}$$

We solving this eigenvalue problem by noting that

$$\det \begin{bmatrix} 1 - \lambda & 1 \\ 3/4 & -\lambda \end{bmatrix} = 0$$

*i.e.*,  $(1 - \lambda)(-\lambda) - 3/4 = 0$  In standard quadratic form this is  $\lambda^2 - \lambda - 3/4 = 0$ . Using the quadratic formula given

$$\lambda = \frac{+1 \pm \sqrt{1 + 4(3/4)}}{2} = 3/2, \text{ or } -1/2$$

For stability any displacement should lead to smaller displacements with time, and this requires that both  $\lambda$ 's be smaller than 1 in magnitude. but one of our  $\lambda$ 's is greater than 1. Therefore some displacements lead to stability and others do not. The stability of the fixed point is ambiguous.

- b) Considering the eigen-condition for  $x$   $x + p = x_1 = \lambda x$ . This implies  $p = (\lambda - 1)x$  or  $x/p = 1/(\lambda - 1)$ . Using  $\lambda = 3/2, -1/2$ , we obtain  $x/p = 2, -2/3$ .