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Linked and knotted beams of light, conservation of helicity and the flow of null electromagnetic fields

William T M Irvine¹

Center for Soft Matter Research, New York University, 4 Washington Place, NY 10003, USA

E-mail: wimirvine@uchicago.edu

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Abstract

Maxwell's equations allow for some remarkable solutions consisting of pulsed beams of light which have linked and knotted field lines. The preservation of the topological structure of the field lines in these solutions has previously been ascribed to the fact that the electric and magnetic helicities, a measure of the degree of linking and knotting between field lines, are conserved. Here we show that the elegant evolution of the field is due to the stricter condition that the electric and magnetic fields be everywhere orthogonal. The field lines then satisfy a 'frozen field' condition and evolve as if they were unbreakable filaments embedded in a fluid. The preservation of the orthogonality of the electric and magnetic field lines is guaranteed for null, shear-free fields such as the ones considered here by a theorem of Robinson. We calculate the flow field of a particular solution and find it to have the form of a Hopf fibration moving at the speed of light in a direction opposite to the propagation of the pulsed light beam, a familiar structure in this type of solution. The difference between smooth evolution of individual field lines and conservation of electric and magnetic helicities is illustrated by considering a further example in which the helicities are conserved, but the field lines are not everywhere orthogonal. The field line configuration at time $t = 0$ corresponds to a nested family of torus knots but unravels upon evolution.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Light in free space provides an ideal playground for the investigation of geometric and topological structures in fields [1–3], with linked and knotted structures receiving growing

¹ Present address: The James Franck Institute and Department of Physics, The University of Chicago, Illinois 60637, USA.

attention of late [4–8]. Examples of loops, knots and links have been considered in different ‘degrees of freedom’ of light fields, such as paraxial vortices, the phase of Riemann–Siberstein vectors and field lines; in this paper we focus on the rules that govern the evolution of linked and knotted field lines.

To motivate and guide our investigation we consider a striking solution to Maxwell’s equations: a beam of light whose electric(magnetic) field lines are all closed loops with any two electric(magnetic) field lines linked to each other. This solution, derived in the context of a topological model for electromagnetism by Rañada [9–12], studied by the author in [8], was recently shown by Besieris and Shaarawi [13] to be equivalent to solutions derived previously by Robinson and Trautman [14, 15], studied and generalized by Bialynicki-Birula [16]. A simple expression for the fields due to Rañada is

$$\mathbf{B} = \frac{1}{4\pi i} \frac{\nabla\eta \times \nabla\bar{\eta}}{(1 + \bar{\eta}\eta)^2}, \quad \mathbf{E} = \frac{1}{4\pi i} \frac{\nabla\zeta \times \nabla\bar{\zeta}}{(1 + \bar{\zeta}\zeta)^2}, \quad (1)$$

$$\zeta(x, y, z, t) = \frac{(Ax + ty) + i(Az + t(A - 1))}{(tx - Ay) + i(A(A - 1) - tz)}, \quad (2)$$

$$\eta(x, y, z, t) = \frac{(Az + t(A - 1)) + i(tx - Ay)}{(Ax + ty) + i(A(A - 1) - tz)}, \quad (3)$$

where $A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1)$, and x, y, z, t are the dimensionless multiples of a length scale a .

At time $t = 0$, the linked field lines are arranged in a structure known as the Hopf fibration [17, 18], a collection of disjoint circles that fill space with the property that any two such circles are linked to each other (see figure 1, blue part). This remarkable structure can be built by first foliating space with tori of different sizes, enclosed inside each other like russian dolls, and subsequently breaking each torus up into a set of circles that wrap once around each circumference of the torus. Since each circle wraps once around each circumference of the torus on whose surface it lives, any two such circles on the same torus will be linked to each other, and, for the same reason, any pair of circles—each from a different torus—will also be linked to each other. To represent such a structure it is convenient to draw the smallest torus (a circle at the center of the nested set), the infinitely big torus (represented by a straight line through the middle of the nested set) and a set of circles from one or two of the intermediate nested tori. Figure 1, shows the electric field lines (orange with smallest/largest torus in red) and the Poynting field lines (gray with smallest/largest torus in black) at time $t = 0$, and representative sets of the electric field lines at subsequent times. Time evolution not only preserves their topological structure but gives them the appearance of filaments that have a continuous identity in time, and evolve by stretching and deforming. Natural questions, addressed here, are *is this indeed true? and if so why?*

The invariance of topological structure in the fields shown in figure 1 has previously been associated [9] with the conservation of electric and magnetic helicities h_e, h_m , which are a measure of the average linking and knotting of electric/magnetic field lines. Using the ‘frozen field’ condition [19] and applying it to this free-space solution, here we point out that although intimately related to conservation of electric or magnetic helicities, what guarantees such a smooth evolution is in fact a more stringent condition, also satisfied by the field, which is that the electric and magnetic fields are everywhere perpendicular $\mathbf{E} \cdot \mathbf{B} = 0$.

Indeed for a free-space field satisfying $\mathbf{E} \cdot \mathbf{B} = 0$, the field lines evolve as if they are stretchable filaments, transported by a velocity field $\mathbf{v} = (\mathbf{E} \times \mathbf{B})/(\mathbf{B} \cdot \mathbf{B})$ [19]. We calculate this velocity field explicitly for the solution of figure 1 and find it to have the form of a Hopf

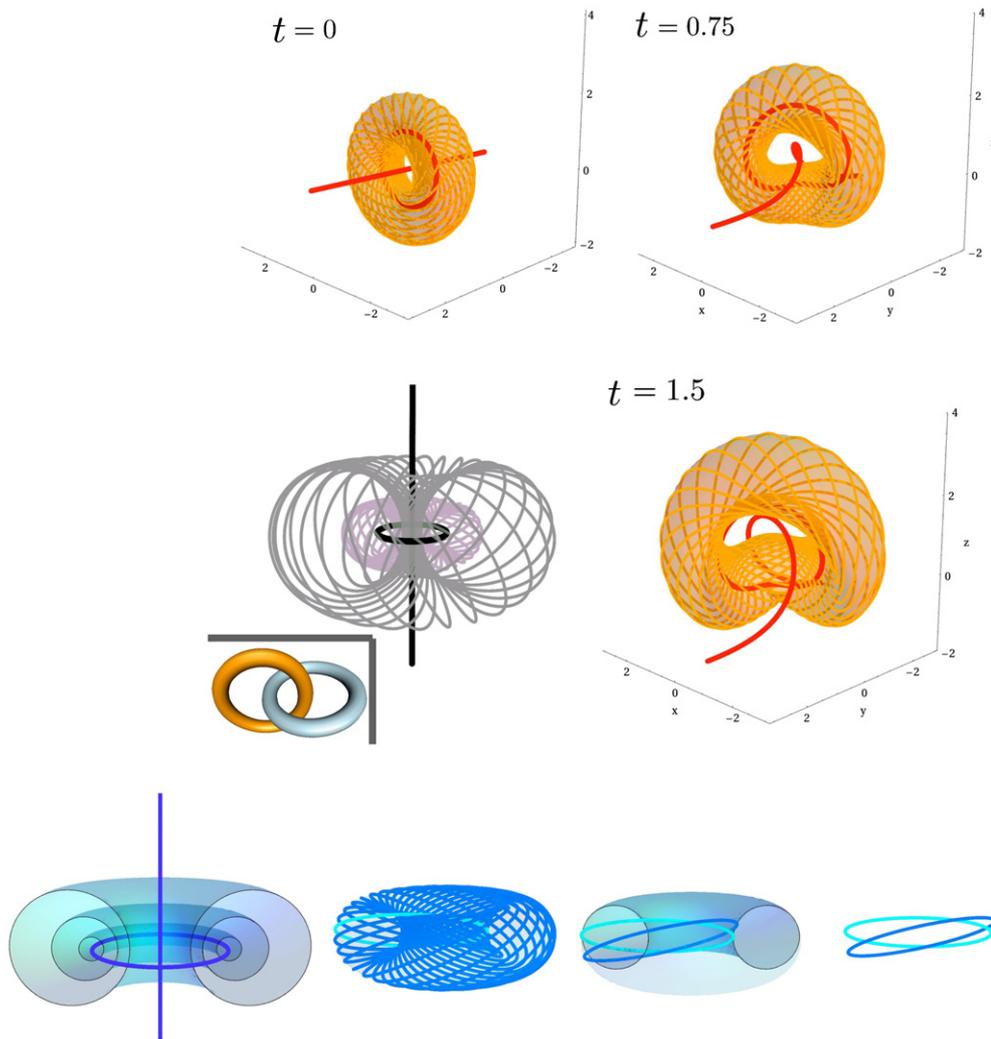


Figure 1. An electromagnetic field with linked field lines. The field lines have the structure of the Hopf fibration, whose construction is shown in blue. The Hopf fibration can be seen as a collection of disjoint circles that fill space with the property that any two such circles are linked to each other. This remarkable structure can be built by first foliating space with tori of different sizes, enclosed inside each other like Russian dolls, and subsequently breaking each torus up into a set of circles that wrap once around each circumference of the torus. Since each circle wraps once around each circumference of the torus on whose surface it lives, any two such circles on the same torus will be linked to each other, and, for the same reason, any pair of circles—each from a different torus—will also be linked to each other. To represent such a structure it is convenient to draw the smallest torus (a circle at the center of the nested set), the infinitely big torus (represented by a straight line through the middle of the nested set) and a set of circles from one or two of the intermediate nested tori. The electric field lines (orange with smallest/largest torus in red) and the Poynting field lines (gray with smallest/largest torus in black) are shown at time $t = 0$, and representative sets of the electric field lines at subsequent times. The electric field lines have the appearance of filaments that have a continuous identity in time and evolve by stretching and deforming, preserving the topological structure of the field.

fibration that, without deformation, (unlike the electric and magnetic field structures which deform) moves along the z -axis at the speed of light. This structure corresponds to the core of the congruence of Robinson [14, 20]. The field is null ($\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{E} \cdot \mathbf{E} = \mathbf{B} \cdot \mathbf{B}$), and the condition $\mathbf{E} \cdot \mathbf{B} = 0$ is guaranteed by the shear-free evolution of the field [14, 21]. To clarify the difference between smooth evolution of individual field lines and conservation of helicity, we consider a striking case of a field whose helicity is conserved, but whose field lines do not satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ everywhere. The field lines that are initially knotted and linked in a striking configuration break up and change topology upon evolution.

2. Electric/magnetic helicity and knottedness

An electromagnetic field has an infinite number of electric(magnetic) field lines. To quantify how linked or knotted they are, it is common to use an average measure; the electric(magnetic) helicity $h_{e(m)}$. h_m is given by

$$h_m = \int d^3x \mathbf{A}(x) \cdot \mathbf{B}(x), \quad (4)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. A similar expression, with \mathbf{B} replaced by \mathbf{E} and \mathbf{A} by a field \mathbf{C} satisfying $\mathbf{E} = \nabla \times \mathbf{C}$ gives the electric field helicity h_e . $h_{e(m)}$ can be understood as an average measure of how much the field lines are knotted and linked [22]. Following Berger [23], we first break up the field into N magnetic flux tubes each with its flux ϕ_i . For each flux tube pair i, j we then compute the linking number L_{ij} . For a pair of flux tubes, L_{ij} measures how linked they are to each other and for a single curve L_{ii} is a measure of knottedness. For example the linking number of the flux tubes shown in figure 1, inset, is 1 whereas the self-linking (Twist+Writhe) of the tube in figure 4, inset, is 3.

L_{ij} can be computed by visual inspection, first projecting the tubes onto a plane and subsequently counting the crossings in an oriented way[24]. Given expressions for a pair of closed lines $\mathbf{c}_1(\tau)$, $\mathbf{c}_2(\tau)$, L_{12} can alternatively be calculated using the Gauss linking integral [23–25]:

$$L(\mathbf{c}_1, \mathbf{c}_2) = \frac{1}{4\pi} \int \frac{d\mathbf{c}_1}{d\tau_1} \cdot \frac{\mathbf{c}_1 - \mathbf{c}_2}{|\mathbf{c}_1 - \mathbf{c}_2|^3} \times \frac{d\mathbf{c}_2}{d\tau_2} d\tau_1 d\tau_2, \quad (5)$$

The flux-weighted average linking between tubes added to the self-linking of each tube is then

$$K = \sum_{i,j=1}^N L_{ij} \phi_i \phi_j. \quad (6)$$

Making the flux tubes finer and finer by letting $N \rightarrow \infty$ and $\phi \rightarrow 0$ it can be shown [23] that K becomes

$$K \rightarrow -\frac{1}{4\pi} \iint d^3x d^3y \mathbf{B}(x) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{B}(y) \quad (7)$$

which using

$$\mathbf{A}(x) = -\frac{1}{4\pi} \int d^3y \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{B}(y), \quad (8)$$

becomes the expression for h_m (equation (4)). Note that $\mathbf{A}(x)$ satisfies $\mathbf{B} = \nabla \times \mathbf{A}$, and can therefore be interpreted as the vector potential.

The time derivative of the magnetic helicity is then given by

$$\partial_t h_m = \partial_t \int d^3x \mathbf{A}(x) \cdot \mathbf{B}(x) \propto \int d^3x \mathbf{E}(x) \cdot \mathbf{B}(x), \quad (9)$$

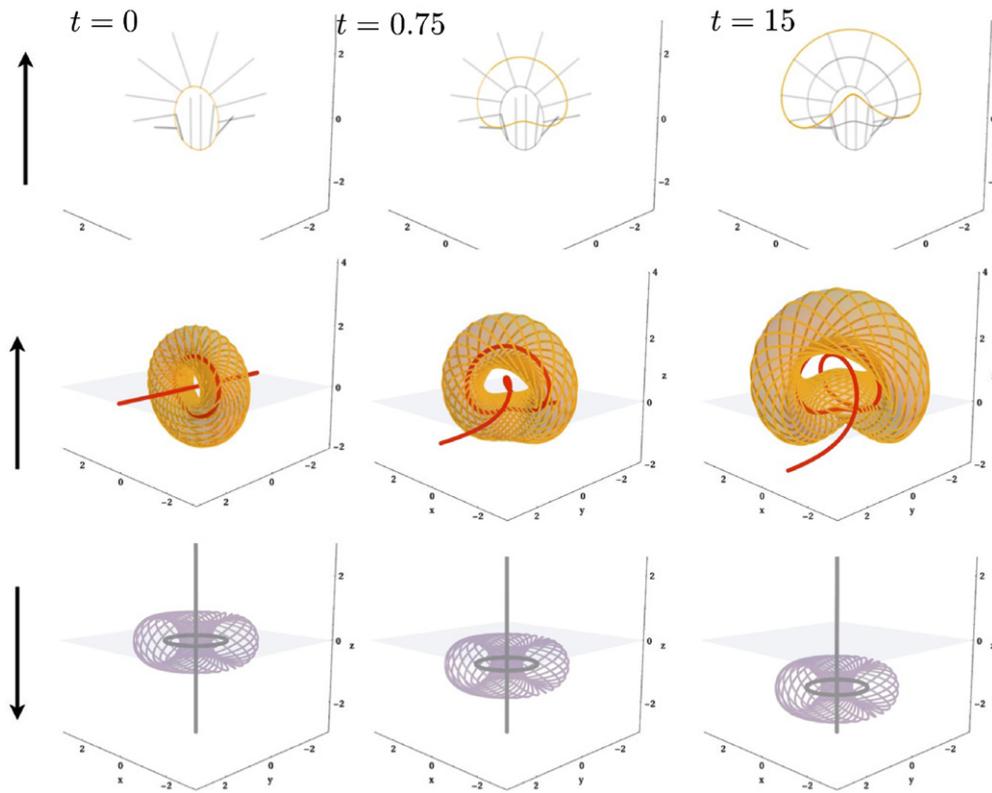


Figure 3. Time evolution of the electric and Poynting field lines. The electric field lines (middle) start out as a Hopf fibration oriented along the x -axis and evolve by translation in the positive z direction accompanied by some twist and distortion. The evolution of the field lines is given by a velocity field whose magnitude is the speed of light and is tangent to the Poynting vector field lines (bottom). Unlike the electric field lines, the Poynting field line structure remains rigid and evolves via a simple translation in a direction *opposite* to the propagation of the field energy. The trajectory taken by each element of any field line (top) is a straight line, tangent to the Poynting field at time $t = 0$.

4. The evolution of a linked null electromagnetic field and its flow field: a congruence of Robinson

We now calculate and study the velocity field of our example null solution. In doing so, we will illustrate not only general aspects of the geometric evolution of a null field, but also develop a vivid picture of the geometry of this particular solution.

The velocity field $v = (E \times B)/(B \cdot B)$ for the Hopf electromagnetic knot of figure 1, is given by

$$v = \frac{1}{(1 + (t + z)^2 + x^2 + y^2)} \begin{pmatrix} 2(x(t + z) - y) \\ 2(y(t + z) + x) \\ 1 + (t + z)^2 - x^2 - y^2 \end{pmatrix}. \quad (12)$$

At time $t = 0$, v is tangent to a Hopf fibration aligned with the z -axis (figure 3). Remarkably, from its functional dependence on $z - t$, v can be immediately seen to evolve by simply moving without change of form along $-z$ (opposite to the energy propagation direction) at the speed

of light (see figure 3). Such a structure is known in the literature as the Robinson congruence [20].

Even though the instantaneous velocity field has this intricate structure, the trajectory of a single element of a field line is remarkably simple: each element travels along a straight line that was tangent to the flow at $t = 0$. This is illustrated in figure 3 in which the straight line trajectories taken by the elements of a field line are shown, with the field line overlaid at different times. It is easy to verify by explicit computation that the field line propagated along such straight paths, at the speed of light, to a new time, is indeed a field line at this new time. In order to determine the field at a later time, it is therefore not necessary to know the form of the velocity field at all times, but only at $t = 0$.

The null condition which ensures the existence of such a propagation by deformation: $\mathbf{E} \cdot \mathbf{B} = 0$, is also satisfied by plane waves, but is not satisfied in general by electromagnetic fields in free space. In particular it is not even a conserved quantity: fields that satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ at time $t = 0$ do not in general satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ at later times. For null fields such as the case under consideration, the preservation of $\mathbf{E} \cdot \mathbf{B} = 0$ is guaranteed if the propagation induced by the velocity field is free of shear [14, 21] (so that the angle between \mathbf{E} and \mathbf{B} does not change). In the language of Robinson, the straight lines along which the field lines propagate correspond to the geodesics of a geodesic shear-free congruence.

Before moving on to examine the role of electric/magnetic helicity conservation in the evolution of the field, we note that in the topological model written by Rañada, the orthogonality condition is also guaranteed within the model if admissible Cauchy data are chosen for the initial value problem[9]. Finally we note in passing that as pointed out in [16], the solution can be obtained from a complex conformal transformation (inversion) on a circularly plane wave, in a way perhaps analogous to the method of generating such a solution proposed in [8] by focusing down to a circularly polarized beam.

5. The evolution of a knotted non-null field with conserved magnetic helicity

Nullness will guarantee continuity of the field lines. To clarify the different roles played by the orthogonality condition on \mathbf{E} and \mathbf{B} , and magnetic/electric helicity conservation, we now consider an example in which the field is not null and magnetic helicity is conserved ($\mathbf{E} \cdot \mathbf{B} \neq 0, \int d^3x \mathbf{E}(x) \cdot \mathbf{B}(x) = 0$).

This example was constructed using a decomposition of the field into vector spherical harmonics found by the author in [8]. In this picture the decomposition of the field studied so far is given by

$$\mathbf{A}(\mathbf{r}, t) = \int dk k^3 e^{-k} \mathbf{A}_{1,1}^-(k, \mathbf{r}) e^{-i\omega t} + \text{c.c.}, \quad (13)$$

where $\mathbf{A}_{1,1}^-(k, \mathbf{r})$ is the Chandrasekhar–Kendall [26] curl and angular momentum eigenstate that satisfies $\nabla \times \mathbf{A}_{l,m}^\pm(k, \mathbf{r}) = \pm k \mathbf{A}_{l,m}^\pm(k, \mathbf{r})^2$.

An example of a field which is non-null but whose helicity is conserved in time is shown in figure 4. The expression for the field is

$$\mathbf{A}(\mathbf{r}, t) = \int dk k^3 e^{-k} e^{-i\omega t} \left[\mathbf{A}_{1,1}^{TE}(k, \mathbf{r}) - i \frac{P}{q} \mathbf{A}_{1,1}^{TM}(k, \mathbf{r}) \right] + \text{c.c.}, \quad (14)$$

where $\mathbf{A}^{TE} = \mathbf{A}^+ + \mathbf{A}^-$ and $\mathbf{A}^{TM} = -i(\mathbf{A}^+ - \mathbf{A}^-)$. At time $t = 0$, the field lines lie on tori that are identical to those of the Hopf fibration, but on each torus the field lines wrap q times

² $\mathbf{A}_{1,1}^\pm(k, \mathbf{r}) = [\mathbf{A}_{1,1}^{TE}(k, \mathbf{r}) \pm i \mathbf{A}_{1,1}^{TM}(k, \mathbf{r})]$, $\mathbf{A}_{lm}^{TE}(k, \mathbf{r}) = \frac{1}{i\omega} f_l(kr) \mathbf{L} Y_{lm}(\theta, \phi)$, $\mathbf{A}_{lm}^{TM}(k, \mathbf{r}) = \frac{1}{k^2} \nabla \times [f_l(kr) \mathbf{L} Y_{lm}(\theta, \phi)]$, $\mathbf{L} = -i\mathbf{r} \times \nabla$ and $f_l(kr)$ is a linear combination of the spherical bessel functions $j_l(kr)$, $n_l(kr)$, determined by boundary conditions. In free-space $f_l(kr) = j_l(kr)/\sqrt{l(l+1)}$.

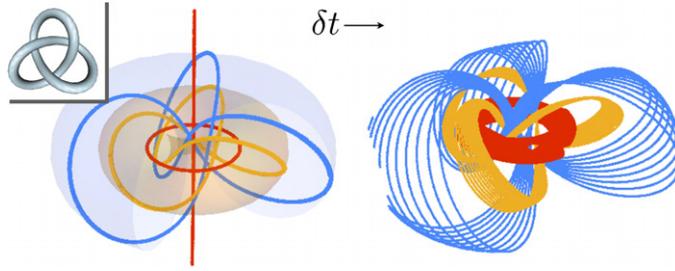


Figure 4. Magnetic field lines of a solution whose magnetic helicity is conserved, but whose electric and magnetic fields are not everywhere perpendicular. The initial structure of the field is that of an infinite set of trefoil knots. Although the average measure of linking and knottedness as measured by the magnetic helicity is conserved, the field lines unravel and do not have a continuous identity in time, providing a striking example of the difference between conservation of magnetic/electric helicity and evolution by smooth deformation.

around the azimuthal direction and p times around the poloidal direction. For $p = 2$ and $q = 3$ they correspond to trefoil knots (figure 4); for general co-prime p, q , they correspond to all possible torus knots [24].

For this solution $\mathbf{E} \cdot \mathbf{B}$ is non-vanishing, but its spatial average is

$$\mathbf{E} \cdot \mathbf{B} \propto \frac{r(r^2 - 1 - t^2) \cos(\theta)}{\pi(r^4 - 2r^2(t^2 - 1) + (1 + t^2)^2)^3}, \quad (15)$$

$$\int d^3x \mathbf{E} \cdot \mathbf{B} = 0. \quad (16)$$

The magnetic helicity is therefore conserved. Although magnetic helicity is conserved, as the solution evolves, the magnetic field lines visibly change topology by unraveling as can clearly be seen in figure 3.

6. Conclusions

To investigate the rules that govern the evolution of field lines in knotted beams of light, we have studied the evolution of the field lines of a free-space solution to Maxwell’s equations, representing a pulsed beam of light whose electric, magnetic and Poynting field lines start out as mutually orthogonal Hopf fibrations [8, 9, 13–16]. We have shown that the electric and magnetic field lines travel and deform upon propagation as if they were filaments embedded in a fluid flowing along the lines of the Poynting vector at the speed of light. We show that this can be done because the field is null and satisfies the ‘frozen field’ condition.

While the electric and magnetic field lines deform upon evolution, those of the flow field, aligned with the Poynting vector, remain rigidly arranged in the form of a Hopf fibration moving along the propagation (z)-axis at the speed of light in the direction *opposite* to the energy flow, a well-known structure (the Robinson congruence) in this type of field [14, 15, 20]. Though the instantaneous flow field has this intricate structure, each segment of the electric and magnetic field lines evolves along a straight line tangent to the flow/Poynting field at time $t = 0$.

We have further examined a solution in which the magnetic/electric helicity, or average knottedness of the field lines, is conserved and whose field lines correspond to an infinite set

of trefoil knots at time $t = 0$. The electric and magnetic fields in this case, however, are not everywhere orthogonal and the field lines break up on evolution. While the null, shear-free condition on a field does not directly relate to knottedness, it appears that it has a key role to play in the evolution of electromagnetic fields with linked and knotted field lines.

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