General:
All lectures will take place on Saturday mornings 11:00AM at the Kersten Physics teaching center room 106. There will be no lecture on Memorial-day weekend. The lecturer, Efi Efrati, is available through email: efrati@uchicago.edu. All handouts and supplementary material will be available at: http://jfi.uchicago.edu/~efrati/compton/index.html

Lecture 1 April 6th 2013

Euclidean and Non-Euclidean geometries

The earliest written text on geometry is an Egyptian papyrus dated to the 2nd millennia B.C. The geometry at that time was a collection of empirically derived principles and formulas devised for application in construction, astronomy and surveying. The latter is where geometry got its name (Greek: geo- "earth", -metron "measurement").

The geometry we know today, also known as Euclidean geometry is derived from the axiomatic formulation put forward by Euclid around 300 B.C. In this formulation Euclid put forward five postulates, which if assumed true, give rise to all the geometry we learned through middle and high-school. The postulates may seem very natural and at times even trivial, yet they cannot be derived logically from any reduced set of assumptions. The postulates read:

1. A straight line may be drawn from any given point to any other.
2. A straight line may be extended to any finite length.
3. A circle may be described with any given point as its center and any distance as its radius.
4. All right angles are equal to one another.
5. Through a given point P not on a line L, there is one and only one line in the plane of P and L which does not meet L. (Playfair's version 1795)

The fifth postulate has a different flavor from first four. Trying to logically derived it from the first four postulates occupied mathematicians for over two thousand years until the 1820's when it was, more or less simultaneously, proved to be indispensable by three different mathematicians working hundreds if not thousands of miles apart: J. Bolyai, N. Lobachevsky and K.F. Gauss. These mathematicians did not try to derive the fifth postulate from the previous four, but alternatively assumed it doesn’t hold. This resulted in perfectly consistent and logical geometries, which are very different from the intuitive and natural Euclidean geometry.

Before we come to describe the properties of these exotic geometries we first note a few corollaries of the five Euclidean postulates:
1. The sum of internal angles in every triangle equals to the sum of two right angles (180 degrees).
2. The Pythagorean theorem: \(a^2 + b^2 = c^2\) for every right-angled triangle, see image to the right.
3. There exists a pair of straight lines that are at constant distance from each other.

In fact, each of the above can substitute Euclid’s fifth postulate and give rise to the same Euclidean geometry. In particular, this implies that none of the above holds in the Non-Euclidean geometries. For example, the sum of internal angles in a triangle not only differs from 180 degrees but also depends on the size of the triangle.

There are two distinct ways to contradict Euclid’s fifth postulate. The first is to claim that there are no parallel lines: Passing through a point \(P\) not on the straight line \(L\), there are no straight lines that never intersect \(L\); this geometry is known as elliptic geometry. Conversely we may postulate that passing through a point \(P\) not on the straight line \(L\) one may find infinitely many straight lines which never intersect \(L\); this geometry is called hyperbolic geometry. A more visual definition of these geometries is as follows. In Euclidean space if two lines are mutually perpendicular to a third straight line, they remain at a constant distance. In hyperbolic geometry straight lines mutually perpendicular to a third straight line draw apart, whereas in elliptic geometry they converge.

The above images imply that straight lines in non-Euclidean spaces behave differently then their Euclidean counterparts. We now come to address how does one define a straight line in such spaces.

The opposite of a straight line is a curved line. We will define a straight line as a line that does not curve. The amount by which a given line in space curves can be formalized as follow: we let a small ant follow the line, we denote the amount of steps the ant takes with its right legs by \(R\), and by \(L\) the number of steps the ant takes with its left legs. The distance between the right and left legs is \(d\). We then define

\[
\text{Turning rate} = \frac{(R-L)/d}{(R+L)/2}
\]

This quantity is also known as the geodesic curvature of the path. If the right legs walk more the turn is to the left. If the left legs walk more the turn is to the right. Only if the right and left legs perform equal amount of steps the trajectory of the ant will have no turning. We call such a line a geodesic line and it is the generalization of a straight line for non-Euclidean geometries.