The geometry of Fractals

Fractals are irregular geometric objects that display features at all scales and often have fractional dimensionality, for which they were named. Fractals can display exact self-similarity, implying that some feature or a portion of the object repeats exactly in a rescaled portion of the object, or approximate self-similarity in which slightly distorted versions of features reoccur at rescaled regions of the object.

One of the key quantitative tools for studying the geometry of a fractal is its fractional (or fractal) dimension. There are many variants of fractal dimension. The first one put forward and perhaps the most popular one, the Hausdorff dimension, quantifies the growth of the number of ball of radius $r$ needed to cover the object of interest as $r$ diminishes to zero. In general the smaller $r$ becomes the more balls are needed to cover the object; if $N(r) \sim 1/r^D$, then the object is said to have Hausdorff dimension $D$ (Due to mathematical subtleties the actual definition is a bit more complex, but relies on the principle detailed above). The Hausdorff dimension of a smooth curve reads one, and for a solid rectangle it reads two. However, more complex objects yield fractional answers, such as the coastline of Britain which has the fractional dimension $\sim 1.25$.

The Mandelbrot set, studied by the mathematician Benoit Mandelbrot, is the set of all values of the complex constant $C$, such that starting from zero repeated iteration of the function $F(z)=z^2+C$, remain bounded. For every $C$ we define $Z_n(C)= (Z_{n-1}(C))^2+C$, and $Z_1(C)=0$. If all the values of $Z_n(C)$ are bounded, $C$ is said to lie in the Mandelbrot set. Visualizing the Mandelbrot set we colored black points inside the set, and color points outside the set according to the rate of divergence of $Z_n(C)$. The images to the right depict the full set (top), and a small detail of the set magnified by a factor of about one million (bottom).

Iterative maps often give rise to fractal structures. However, fractal structures appear also in natural growth, in diffusion processes, in electric discharges as well as in terrestrial topography. In all these examples the resolution is finite, thus the refinement of objects does not continue indefinitely. Nonetheless, they display fractal properties over a wide range of scales.

The irregular geometry of fractals make the standard geometric tools such as measuring of lengths and areas and the definition of tangents impossible, and new tools such as the fractional dimensionality are needed for their analysis.