The 77th Compton lecture series
Frustrating geometry:
Geometry and incompatibility shaping the physical world

Lecture 8:
Phyllotaxis, the golden ratio
and the Fibonacci sequence

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The University of Chicago
The golden ratio: Myth vs. Reality

It appears naturally in many systems:
Arrangement of leaves, the nautilus shell, the symmetry of our body

It is the most eye pleasing aspect ratio:
most appealing rectangle, used by architects (Parthenon, Le Corbusier)

It is related to the Fibonacci sequence
It is the most irrational number
The golden ratio: Myth vs. Reality
Outline:

• What is the golden ratio
• What is the Fibonacci sequence
• Phyllotactic patterns

• The golden ratio, Fibonacci sequence and continued fractions
• A physicist version of Phylotaxis
• Truth and myth of the golden ratio
The golden ratio

\[ \frac{a + b}{a} = \frac{a}{b} = \Phi \]

\[ \Phi = \frac{1 + \sqrt{5}}{2} = 1.618033988... \]
The Fibonacci sequence

Liber Abaci - The book of calculation

Leonardo of Pisa (Fibonacci)

DCCCCXXXVIIII = 949
The Fibonacci sequence

Rabbit population growth rule:

A newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?
### The Fibonacci sequence

<table>
<thead>
<tr>
<th>Generation</th>
<th>Non-Mature</th>
<th>Mature</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>Fn-1</td>
<td>Fn</td>
</tr>
</tbody>
</table>

\[ F_{n+1} = F_{n-1} + F_n \]

- \( \frac{F_n+1}{F_n} \rightarrow \Phi \)
- \( \Phi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \)

<table>
<thead>
<tr>
<th>Generation</th>
<th>Non-Mature</th>
<th>Mature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
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<tr>
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<td>6</td>
<td>5</td>
<td>8</td>
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<td>7</td>
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<td>21</td>
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<tr>
<td>9</td>
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<td>10</td>
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<td>11</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>12</td>
<td>89</td>
<td>144</td>
</tr>
</tbody>
</table>
The Fibonacci sequence

\[ F_{n+1} = F_{n-1} + F_n \]

Fibonacci soup...

Recipe of Today’s soup:
1/2 yesterday’s soup + 1/2 the day before yesterday’s soup
Phyllotaxis

Phyllotaxis: The geometric arrangement of leaves on a plant stem
Phyllotaxis

Phyllotaxis: The geometric arrangement of leaves on a plant stem
Phyllotaxis: The geometric arrangement of leaves on a plant stem.
Phyllotaxis

Phyllotaxis: The geometric arrangement of leaves on a plant stem

8 and 13
Phyllotaxis

21 and 34
Phyllotactic spirals

34 and 55

photograph S. Morris
Phyllotactic spirals
Phyllotactic spirals

Why and How?
Reasoning for Phyllotactic patterns

Any rational angle will eventually lead to vertical shadowing of leaves

A plant of a finite length cannot tell the difference between $\pi$ and $3141592/1000000$

Are some irrational numbers more irrational than others?
Can you tell this by examining only finite fractions?
Continued fractions, Diophantine approximation and Hurwitz's theorem

Diophantine approximations deal with the approximations of a real numbers by rational numbers.

\[ \left| \alpha - \frac{p}{q} \right| \leq \left| \alpha - \frac{p'}{q'} \right|, \quad \text{for all } 0 < q' \leq q. \]

Hurwitz’s theorem: For every irrational \( \alpha \) there are infinitely many integer pairs \( p, q \) such that

\[ \left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}. \]
Continued fractions, Diophantine approximation and Hurwitz's theorem

\[
3 + \frac{1}{7} = \frac{22}{7} \approx 3.1428
\]
\[
3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} \approx 3.1415094
\]
\[
\frac{1}{7} \approx 0.14
\]
\[
\frac{1}{106} \approx 0.0094
\]
\[
\pi - \frac{22}{7} \approx -0.0012
\]
\[
\pi - \frac{333}{106} \approx 0.0000832
\]
Continued fractions, Diophantine approximation and Hurwitz's theorem

Hurwitz’s theorem is optimal

\[ \left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}. \]
Continued fractions, Diophantine approximation and Hurwitz's theorem

\[
\frac{415}{93} = 4 + \frac{1}{2 + \frac{1}{6 + \frac{1}{7 + \frac{1}{\ddots + \frac{1}{a_n}}}}}
\]

Finite truncations of the continuous fraction gives its convergents.
Continued fractions, Diophantine approximation and Hurwitz's theorem

according to how difficult they are to approximate with rationals. It is in this sense that one irrational is more irrational than another. To make the criterion precise, we start from the following fact:

Hurwitz' Theorem: Every number has infinitely many rational approximations \( \frac{p}{q} \), where the approximation \( \frac{p}{q} \) has error less than \( \frac{1}{q^2} \).

The criterion can then be stated in terms of: how much less than \( \frac{1}{q^2} \)? to compare and in this regard, some of the best rational approximations can be displayed in a table. Here we will reckon the error \( E \) in terms of Hurwitz' bound \( M = \frac{1}{q^2} \) by tabulating the quotient \( E/M \).

<table>
<thead>
<tr>
<th>( p/q )</th>
<th>( E = \text{error} )</th>
<th>( M = \frac{1}{\sqrt{5}q^2} )</th>
<th>( E/M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>22/7</td>
<td>.0126</td>
<td>.091</td>
<td>0.13</td>
</tr>
<tr>
<td>355/113</td>
<td>.000000266</td>
<td>.0000350</td>
<td>0.007</td>
</tr>
<tr>
<td>( \sqrt{2} ):</td>
<td>( p/q )</td>
<td>( E = \text{error} )</td>
<td>( M = \frac{1}{\sqrt{5}q^2} )</td>
</tr>
<tr>
<td>7/5</td>
<td>.0142</td>
<td>.0179</td>
<td>0.79</td>
</tr>
<tr>
<td>239/169</td>
<td>.0000124</td>
<td>.0000156</td>
<td>0.79</td>
</tr>
</tbody>
</table>

These tables suggest that \( e \) admits much better rational approximations than \( \pi \). In fact no rational approximation to \( e \) ever gets an \( E/M \) ratio as small as 0.13, let alone 0.007, and \( e \) is really harder to approximate with rationals than \( \pi \). In this precise sense \( e \) is a more irrational number than \( \pi \).
Continued fractions, Diophantine approximation and Hurwitz's theorem

According to how difficult they are to approximate with rationals. It is in this sense that one irrational is more irrational than another. To make the criterion precise, we start from the following fact:

Hurwitz' Theorem: Every number has infinitely many rational approximations $p/q$, where the approximation $p/q$ has error less than $1/q^2$.

The criterion can then be stated in terms of: how much less than $1/q^2$? To compare and in this regard, some of the best rational approximations can be displayed in a table. Here we will reckon the error $E$ in terms of Hurwitz' bound $M = 1/q^2$ by tabulating the quotient $E/M$.

<table>
<thead>
<tr>
<th>$\pi$: p/q</th>
<th>E = error</th>
<th>$M = 1/\sqrt{5}q^2$</th>
<th>$E/M$</th>
</tr>
</thead>
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<tr>
<td>22/7</td>
<td>0.0126</td>
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<table>
<thead>
<tr>
<th>$\sqrt{2}$: p/q</th>
<th>E = error</th>
<th>$M = 1/\sqrt{5}q^2$</th>
<th>$E/M$</th>
</tr>
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<tbody>
<tr>
<td>7/5</td>
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</tbody>
</table>

These tables suggest that $\pi$ admits much better rational approximations than $\sqrt{2}$. In fact no rational approximation to $\pi$ ever gets an $E/M$ ratio as small as 0.13, let alone 0.007, and $\pi$ is really harder to approximate with rationals than $\sqrt{2}$. In this precise sense $\pi$ is a more irrational number than $\sqrt{2}$. On to next irrational page.

Back to first irrational page.

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Irrational Number 2

http://www.math.sunysb.edu/~tony/whatsnew/column/irrational...
The most irrational number

\[ \Phi = \frac{1 + \sqrt{5}}{2} \]

<table>
<thead>
<tr>
<th>convergent</th>
<th>E/M</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_1 = 1/1</td>
<td>1.382</td>
</tr>
<tr>
<td>c_2 = 2/1</td>
<td>0.8541</td>
</tr>
<tr>
<td>c_3 = 3/2</td>
<td>1.055</td>
</tr>
<tr>
<td>c_4 = 5/3</td>
<td>0.9787</td>
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<tr>
<td>c_5 = 8/5</td>
<td>1.008</td>
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<tr>
<td>c_6 = 13/8</td>
<td>0.9968</td>
</tr>
<tr>
<td>c_7 = 21/13</td>
<td>1.001</td>
</tr>
<tr>
<td>c_8 = 34/21</td>
<td>0.9995</td>
</tr>
</tbody>
</table>

\[ 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

\[ \left(1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \right) \]

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The Phyllotactic spiral

The distribution of new leaves at an angular separation equal to the golden angle gives the most uniform angular distribution.

How?

credit: Rolf Rutishauser & Jacques Dumais
A physicist’s Phyllotactic spiral

The distribution of new leaves at an angular separation equal to the golden angle gives the most uniform angular distribution.

Phyllotaxis as a Physical Self-Organized Growth Process

S. Douady\(^{(a)}\) and Y. Couder

New leaves grow at some definite angles governed by their interactions with existing leaves, and are then advected away
A physicist’s Phyllotactic spiral

The distribution of new leaves at an angular separation equal to the golden angle gives the most uniform angular distribution.

**Phyllotaxis as a Physical Self-Organized Growth Process**

S. Douady\(^{(a)}\) and Y. Couder
A physicist’s Phyllotactic spiral

The distribution of new leaves at an angular separation equal to the golden angle gives the most uniform angular distribution.

Phyllotaxis as a Physical Self-Organized Growth Process

S. Douady(a) and Y. Couder
A physicist’s Phyllotactic spiral

The distribution of new leaves at an angular separation equal to the golden angle gives the most uniform angular distribution.

credit: Richard Smith
Now what about the quasi-crystal?
Now what about the quasi-crystal?
The golden ratio introduced by man
The logarithmic spiral

\[ r = ae^{b\theta} \]
Summary

The golden ratio truly is a remarkable number

When coming to examine a phenomena, quantitative objectivity is essential.

Further reading: