The Renormalization Revolution

Leo P. Kadanoff
The University of Chicago
Chicago, Illinois, USA
and
The Perimeter Institute
Waterloo, Ontario, Canada

email: LeoP@UChicago.edu

1

abstract

In present-day physics, the renormalization method, as developed by Kenneth G. Wilson, serves as the primary means for constructing the connections between theories at different length scales. This method is rooted in both particle physics and the theory of phase transitions. It was developed to supplement mean field theories like those developed by van der Waals and Maxwell, followed by Landau.

Sharp phase transitions are necessarily connected with singularities in statistical mechanics, which in turn require infinite systems for their realization. (I call this result the extended singularity theorem.) A discussion of this point apparently marked a 1937 meeting in Amsterdam celebrating van der Waals.

Mean field theories neither demand nor employ spatial infinities in their descriptions of phase transitions. Another theory is required that weds a breaking of internal symmetries with a proper description of spatial infinities. The renormalization (semi-)group provides such a wedding. Its nature is described. The major ideas surrounding this point of view are described including especially scaling, universality, and the development of connections among different theories.

Connections in Particle Physics

Particle physics often wishes to relate its present, phenomenological, theory to a deeper (?!) theory at a much shorter or longer length scale. e.g. Connect the standard model to physics at a LHC, unification, or Planck scale.

Previously the search for a final theory has been impeded by ugliness or singularities arising at scales far from observation. These singularities show up directly as infinities in perturbation theory and indirectly as algebraic behavior ($1/|x-y|^p$) in correlation functions

5

Connections in Condensed Matter Physics

Condensed matter physics relates the observable, often macroscopic, properties of liquids, gases, solids and all everyday materials to more microscopic theories, often the quantum theory of atoms and molecules. Since the macroscopic theories are themselves non-trivial, e.g. elasticity, hydrodynamics, the electrodynamics of materials, it follows that condensed matter physics is largely an exercise in connecting different kinds of theories.

Typically this connection involves different length scales

Size of molecule = 10^{-9} meter. Size of laboratory= 5 meter

4

Connections and Differences in Statistical Mechanics

One of the deepest aspects of this area of science is the existence of different thermodynamic phases, each with qualitatively different properties. E.g., freezing is a sudden qualitative change in which the material abruptly gains rigidity. How can this happen? How can there be qualitative distinctions among different phases of matter.

I followed statistical and condensed matter physics in most of these lectures, but particle physics and condensed matter physics are closely connected.

Further Connections

Field Theory and Statistical Mechanics are closely connected. A Wick rotation $t \rightarrow i/(kT)$ will take you from one to the other.

Quantum Mechanics and Classical Mechanics are closely connected. Both employ Hamiltonians as basic generators of time development as do Field Theory and Statistical Mechanics.

All four have a dual structure in which terms in the Hamiltonian both describe measurable quantities and equally generate changes in development.

All four have the same structure: Poisson Bracket and Commutator, conjugate variables = p's and q's.

I shall talk mostly about statistical mechanics.

6

Further Connections

Field Theory and Statistical Mechanics are closely connected. A Wick rotation $t \to i/(kT)$ will take you from one to the other.

Quantum Mechanics and Classical Mechanics are closely connected. Both employ Hamiltonians as basic generators of time development as do Field Theory and Statistical Mechanics.

All four have a dual structure in which terms in the Hamiltonian both describe measurable quantities and equally generate changes in development.

All four have the same structure: Poisson Bracket and Commutator, conjugate variables = p's and q's.

I shall talk mostly about statistical mechanics.

Dirac's ideas

6

Context

This lecture is in part taken from the Dirac Lecture that I gave in St. Johns College Cambridge, Dirac's college.

"I might have thought that the new ideas were correct if they had not been so ugly"
Dyson quoting Dirac on renormalization.

Kramers, already mentioned, and Dirac were probably the first inventors of renormalization methods. These methods were extensively used to go around problems of infinities in field theories, classical and quantum.

Perhaps the first use of this kind was by Boltzmann who used $E=3PV$ for a photon gas to derive Stephan-Boltzmann law ($E \sim T^4$), knowing full well that both E and P were infinite according to the statistical mechanics of his day.

For mean field theory

the physics is an averages of order parameters and statistical quantities. But near phase transitions mostly,

The physics is in fluctuations

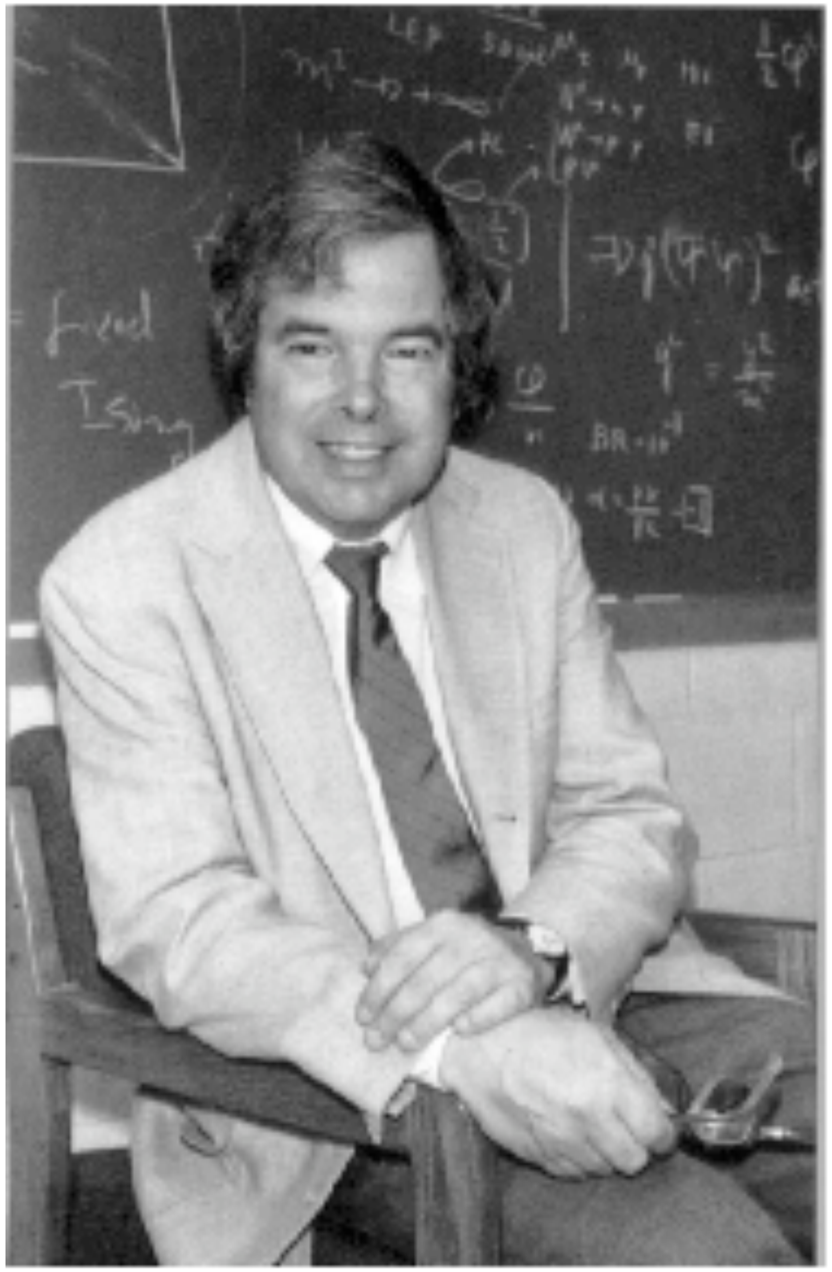
which extend over an indefinite range at critical point. t and h limit range of fluctuations to finite value, called the correlation length, ξ. How can we convert this fact into a theory?

At the singularities these fluctuations are droplets of fluid which have all different scales from the microscopic to as large as you want. Away from singularity correlation length serves to cut off the largest-scale fluctuations. These droplets are regions of density different from that of the surrounding fluid.

The Renormalization Revolution:

precursors:

- Onsager solves d=2 Ising model. His results disagree with mean field theory.
- King's College School (Cyril Domb, Martin Sykes, Michael Fisher) do expansions in K and exp(-K) and find mean field theory critical indices are wrong.
- Patashnskii & Pokrovsky look at correlations in fluctuations
- Benjamin Widom gets scaling and phenomenology right
- Kadanoff suggests partial direction of argument



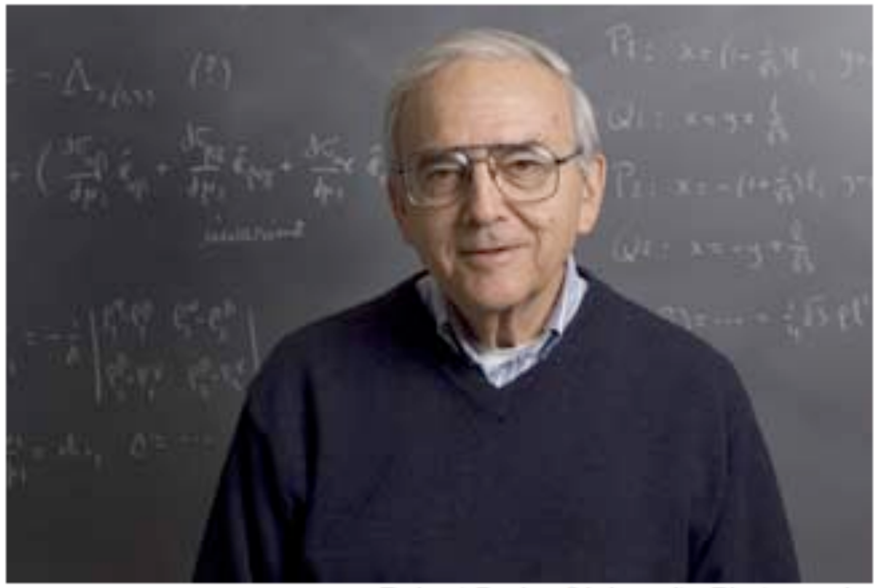
Kenneth G. Wilson synthesizes new theory

Toward the revolution

The phenomenology

Ben Widom noticed the most significant scaling properties of critical phenomena, but did not detail where they might have come from.

B. Widom: J Chem. Phys. **41** 1643 (1964)
B. Widom, J. Chem. Phys. **43** 3892 and 3896 (1965).



Robert Barker/University Photography
Professor Benjamin Widom in his office in Baker Lab.

Widom's results

in terms of $t=T-T_c$ $h=p-p_c$

Widom 1965: scaling result He focuses attention on scaling near critical point. In this region, averages and fluctuations have a characteristic size, for example density jump ~ $(-t)^{\beta}$ when h=0

density minus critical density ~ $(h)^{1/\delta}$ when t=0

Therefore, Widom argues there is a characteristic size for h, which is

$h^* \sim (-t)^{\beta\delta} = (-t)^{\Delta}$ with $\Delta=\beta\delta$

so that density minus critical density = $(-t)^{\beta} g(h/t^{\Delta})$

therefore, using a little thermodynamics, scaling for free energy is

$F(t,h)= V\, t^{\beta+\Delta} f^*(h/t^{\Delta})+ F_{non\text{-}singular}$: (V is volume of system)

Further he says singular term in free energy given by excitations of size of coherence length with kT per excitation. They fill all space, giving

$F - F_{non\text{-}singular} \sim$ (Volume of system)/ $\xi^d \sim V t^{d\nu}$

Therefore "magic" relations, e.g. $\beta + \Delta = d\,\nu$

additional phenomenology

Pokrovsky & Patashinskii study correlation functions, build upon Widom's work

A.Z. Patashinskii and V.L. Pokrovsky ",Soviet Phys. JETP, **19** 667(1964)

They have scaling ideas $\sigma(r) \sim 1/r^x$ so they predict
$< \sigma(r_1)\, \sigma(r_2) \ldots\ldots \sigma(r_m) > \sim 1/r^{mx}$

Then if $\xi \sim (t)^\nu$ one could argue that $< \sigma(r_1) > \sim (-t)^\beta$ so $\beta = x\,\nu$

one could also argue that $\int dr_1 < [\sigma(r_1) - <\sigma>]\,[\sigma(r_2) - <\sigma>] >= d<\sigma>/dh$

$\sim (-t)^\gamma \sim (-t)^{-2x\,\nu + d\nu}$ so $\gamma = d\nu - 2\,\beta$

After Widom, Michael E. Fisher introduces scaling ideas, and two basic indices (like Widom) in his 1965 paper in the University of Kentucky conference on phase transitions. He bases his approach upon an insightful view of droplets of the different phases driving the thermodynamics. However, he misses the relation between correlation length and thermodynamics.

Kadanoff (1966) .....

There were also correlation function calculations performed that could suggest scaling

e.g. LPK and coworkers (including Robert Hecht) (1965-1972) for d=2 Ising model, using Onsager solution:

$<[\sigma(\mathbf{r})-<\sigma>]\,[\sigma(\mathbf{s})-<\sigma>]=|\,\mathbf{r}-\mathbf{s}\,|^{-1/4}\;H(|\mathbf{r}-\mathbf{s}|/\xi\,)$

$\xi \sim a\,|t|^{-1}$. At h=0

$<\sigma>=\,<\varepsilon>=0$ at criticality ε is energy density $\sigma(\mathbf{r})\sigma(\mathbf{s})$ (n.n.)

$<\sigma(\mathbf{r})\sigma(\mathbf{s})> \sim |\,\mathbf{r}-\mathbf{s}\,|^{-1/4}$ at criticality

$<\varepsilon(\mathbf{r})\varepsilon(\mathbf{s})> \sim |\,\mathbf{r}-\mathbf{s}\,|^{-2}$ at criticality,

$<[\varepsilon(\mathbf{r})-<\varepsilon>]\,[\sigma(\mathbf{s})-<\sigma>]=|\,\mathbf{r}-\mathbf{s}\,|^{-9/8}\;J(|\mathbf{r}-\mathbf{s}|/\xi\,)$ At h=0

$<\varepsilon(\mathbf{r})\;\sigma(\mathbf{s})\;\sigma(\mathbf{t})>=\tfrac{1}{2}\,|\mathbf{s}-\mathbf{t}|^{3/4}\;[|\mathbf{r}-\mathbf{s}|\;|\mathbf{r}-\mathbf{t}|]^{-1}$ at criticality

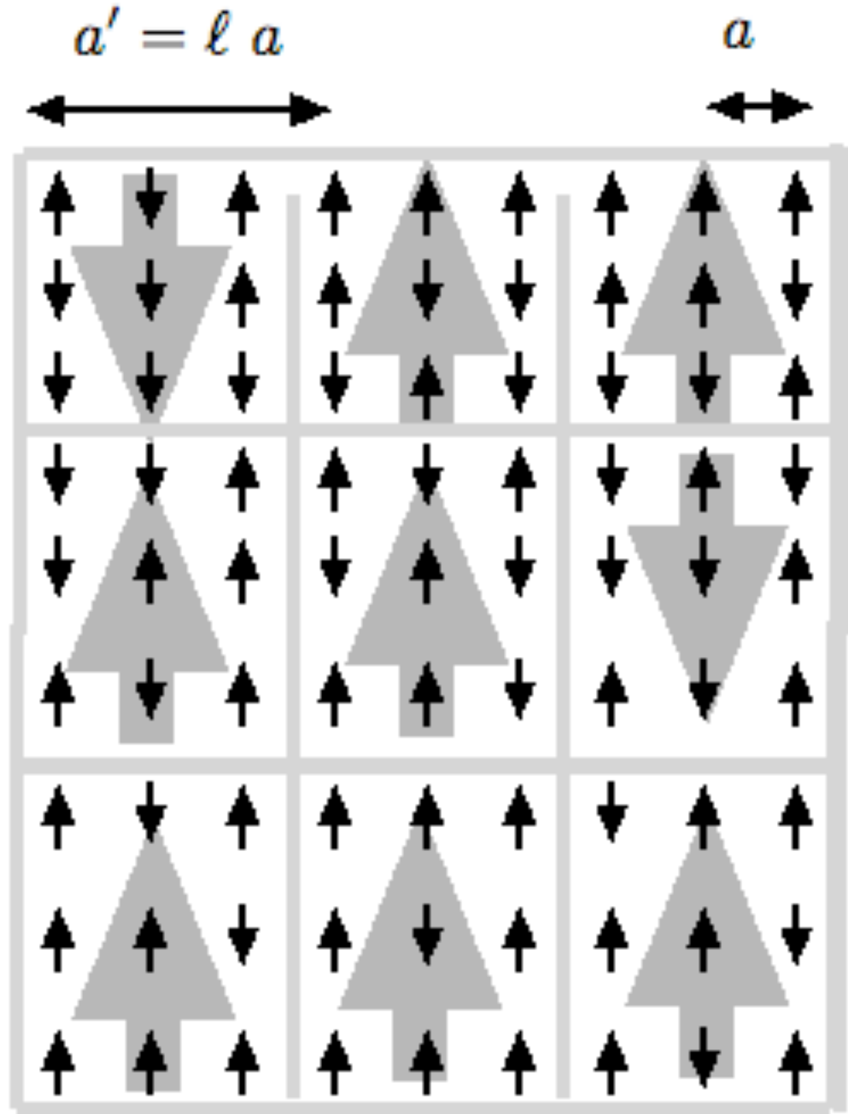
Additivity of indices is not an accident, explained by $\sigma \sim r^{-x}$ $x=x_\sigma=1/8$

ε scales with index $x_\varepsilon=1$ Existence of a single correlation length will turn out to be important also.

Block Scaling 1966

Kadanoff considers invariance properties of critical point and asks how description might change if one replaced a block of spins by a single spin, thus changing the length scale and having fewer degrees of freedom.

Answer: There are new effective values of $(T-T_c)=t$, $(p-p_c)=h$, and free energy per spin K_0. These describe the system just as well as the old values. Fewer degrees of freedom imply new couplings, but no change at all in the physics. This result incorporates both scale-invariance and universality.



$$N' = N/\ell^d$$

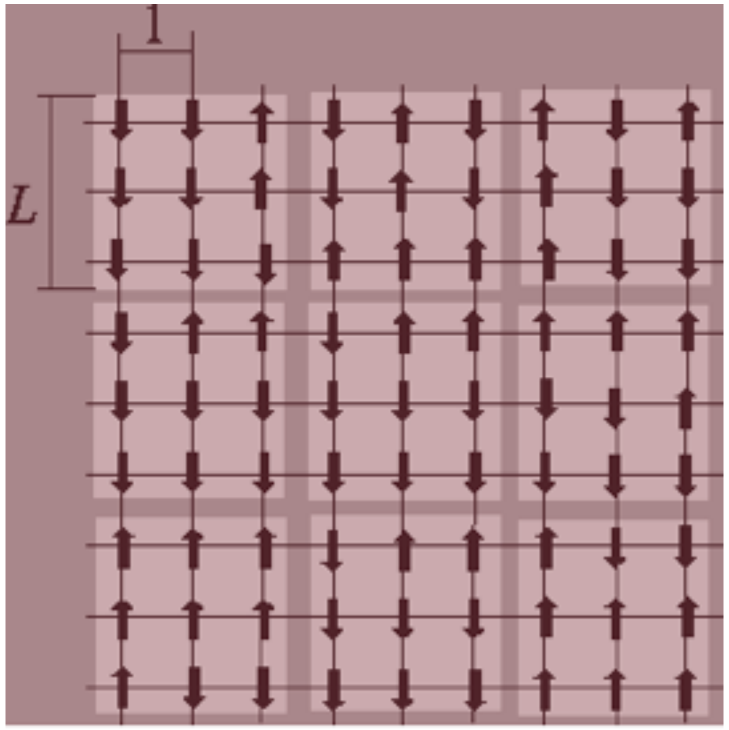
$$h' = h\ \ell^{y_h}$$

$$t' = t\ \ell^{y_t}$$

Renormalization for Ising model in any dimension; reprise

$Z = \text{Trace}_{\{\sigma\}} \exp(W_K\{\sigma\})$

Each box in the picture has in it a variable called μ_R, where the R's are a set of new lattice sites with nearest neighbor separation 3a. Each new variable is tied to an old ones via a renormalization matrix $G\{\mu, \sigma\} = \prod_R g(\mu_R, \{\sigma\})$ where g couples the μ_R to the σ's in the corresponding box. We take each μ_R to be ±1 and define g so that, $\Sigma_\mu\, g(\mu,\{s\}) = 1$.



fewer degrees of freedom produces "block renormalization"

Now we are ready. Define $\exp(W'\{\mu\}) = \text{Trace}_{\{\sigma\}}\, G\{\mu, \sigma\} \exp(W_K\{\sigma\})$

Notice that $Z = \text{Trace}_{\{\mu\}} \exp(W'\{\mu\})$

If we could ask our fairy god-mother what we wished for now, it would be that we came back to the same problem as we had at the beginning: $W'\{\mu\} = W_{K'}\{\mu\}$ where the subscript represents the three relevant couplings in the system: t , h, K_0.

Renormalization of couplings

Transformations: $a \dashrightarrow \ell\, a = a'$ $\quad W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $\quad Z' = Z$ $\quad \mathbf{K}' = R(\mathbf{K})$

Scale Invariance at the critical point: --> $K_c = R(K_c)$
Temperature Deviation: $K = K_c - t$ $\quad K' = K_c - t'$
critical point: if $t=0$ then $t'=0$

coexistence (ordered) region: if $(t<0, h=0)$ then $(t'<0, h'=0)$

disordered region $(t>0, h=0)$ goes into disordered region $(t'>0, h=0)$

if t is small, $t'=bt$. $\quad b=(a'/a)^y$ defines $y=y_t$ = critical index for temperature. $b>1$ implies motion away from critical point

near critical point, $h'=b_h h$ $\quad \ln b_h = y_h \ln (a'/a)$ defines y_h, which then describes renormalization of magnetic field or pressure.

b's can be found through a numerical calculation.

In this context, the couplings are called *fields* indicating that they can vary from point to point. The other basic quantities in the theory are called operators (e.g. $\varepsilon(\mathbf{r})$ *and* $\sigma(\mathbf{s})$). These can be renormalized also

Other Renormalizations

coherence length: $\xi=\xi_0\, a\, t^{-\nu}$ 2d Ising has $\nu=1$; 3d has $\nu\approx 0.64$....

$\xi=\xi'$ $\quad\quad \xi_0\, a\, t^{-\nu} = \xi_0\, a'\, (t')^{-\nu}$

so $\nu=1/y=1/y_t$ (there are other y's going with other fields)

number of lattice sites: $N=\Omega/a^d$ $\quad N'=\Omega/a'^d$

$N'/N= a^d / a'^d = (a'/a)^{-d}$

Free energy: $F = ...+Nf_c(t)= F'= ...+N'fc(t')$

$f_c(t) = f_c^0\, t^{d\nu}$

Specific heat: $C=d^2F/dt^2\sim t^{d\nu-2} = t^{-\alpha}$ form of singularity determined by y:

$2-\alpha = d\,\nu$

Renormalization of operators

Transformations: $a \dashrightarrow 3a = a'$ $\quad W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $\quad Z' = Z$ $\qquad \mathbf{K}' = R(\mathbf{K})$

$$W_K\{\sigma\} = W_{K_c}\{\sigma\} + \sum_{\mathbf{r}} [t\, \varepsilon(\mathbf{r}) + h\, m(\mathbf{r})]$$

$$= W_{K_c}\{\mu\} + \sum_{\mathbf{R}} [\, t'\, \varepsilon'(\mathbf{R}) + h' m'(\mathbf{R})\,]$$

critical point "free energies", W are identical so

$$\sum_{\mathbf{r}} [t\, \varepsilon(\mathbf{r}) + h\, m(\mathbf{r})] = \sum_{\mathbf{R}} [\, t'\, \varepsilon'(\mathbf{R}) + h' m'(\mathbf{R})\,] \quad \text{so}$$

$$\sum_{\mathbf{R}} h' m'(\mathbf{R}) = \sum_{\mathbf{r}} [h\, m(\mathbf{r})]$$

renormalization gives $h' = b_h h$ $\quad \ln b_h = y_h \ln (a'/a) = = y_h \ln \ell$; $\quad \sum_{\mathbf{r}} = \ell^d \sum_{\mathbf{R}}$

Consequently, $m'(\mathbf{R}) = m(\mathbf{r})\ \ell^x$ $\quad$ or $\mu(\mathbf{R}) = \sigma(\mathbf{r})\ \ell^x$ $\qquad$ with $x = d - y_h$

In fact, any congugate pair of operator/field have indices *x* and *y* obeying the *hyperscaling* relation $x+y=d$.

Ising model $d=2$, $x_\sigma = 1/8$; $x_\varepsilon = 1$; $y_h = 15/8$; $x_t = 1$

Fields: Relevant, Irrelevant, marginal, ...

a field is a number multiplying a possible term in the Hamiltonian

there are a few relevant fields: like t,h, K_0, which grow at larger length scales, completely dominate large-scale behavior

there are many irrelevant fields: they have |b|<1, don't play a role at large scales

and also many redundant fields: they drop out of free energy calculation

there can be but usually are not marginal fields. they have |b|=1 and produce a continuously varying behavior at critical point

Universality: marginal variables are rare, mostly problems fall into a few universality classes. All problems in a given class have the same fixed point and the same critical and near-critical behavior.

Universality classes:
Ising ferromagnets+all single-axis ferromagnets + all long-gas phase transitions (Z2 symmetry)
Heisenberg Model ferromagnet (U3 symmetry)
superconductor, superfluid, easy plane ferromagnet (U2 symmetry).

A Worthwhile Phenomenology

Weaknesses of phenomenology:

i. One cannot calculate everything: values of y's are unknown

ii. One cannot be sure about what parts of theory are right, what parts wrong.

iii. cannot determine universality classes from theory

iv. Does not provide lots of indication of what one should do to make the next step.

The physics is in fluctuations

which extend over an indefinite range at critical point. t and h limit range (called the correlation length,ξ) to finite value

As renormalization is done, the lattice constant assumes a new value a´= ℓa

The new deviation from the critical temperature is $t' = t\ \ell^{y_t}$

The new pressure variable is $h' = h\ \ell^{y_h}$

but the coherence length is just the same.

Since the length scale is irrelevant h and t must appear in the combination $h/t^{y_h/y_t}$ while the coherence length appears as $a/t^{1/y_t}$ which is invariant. The demand that the ℓ cancels out of all physical results produces the phenomenology of Widom.

Start Here

Monday	Wednesday	Friday
1 /3 class	1/5 class	1/7 class
1/10 class	1/12 class	1/14 Arnab
1/17 day off	1/19 class	1/21
1/24 class (homework 3 due)	1/26 class (4-5)	1/28 class (5)
1/31 class (5)	2/2 no class	2/4 class (5)
2/7 class (6) (homework 5 due)	2/9 day off	2/11 class (6)
2/14 class (6)	2/16 (6)	2/18 class (4)
2/21 class (7)	2/23 (7) (homework 6 due)	2/25 class (7/8)
2/28 class (8)	3/2 (8/9)	3/4 class(9)
3/7 (9) (homework 7 due)	3/9 (complexity lecture?)	no class

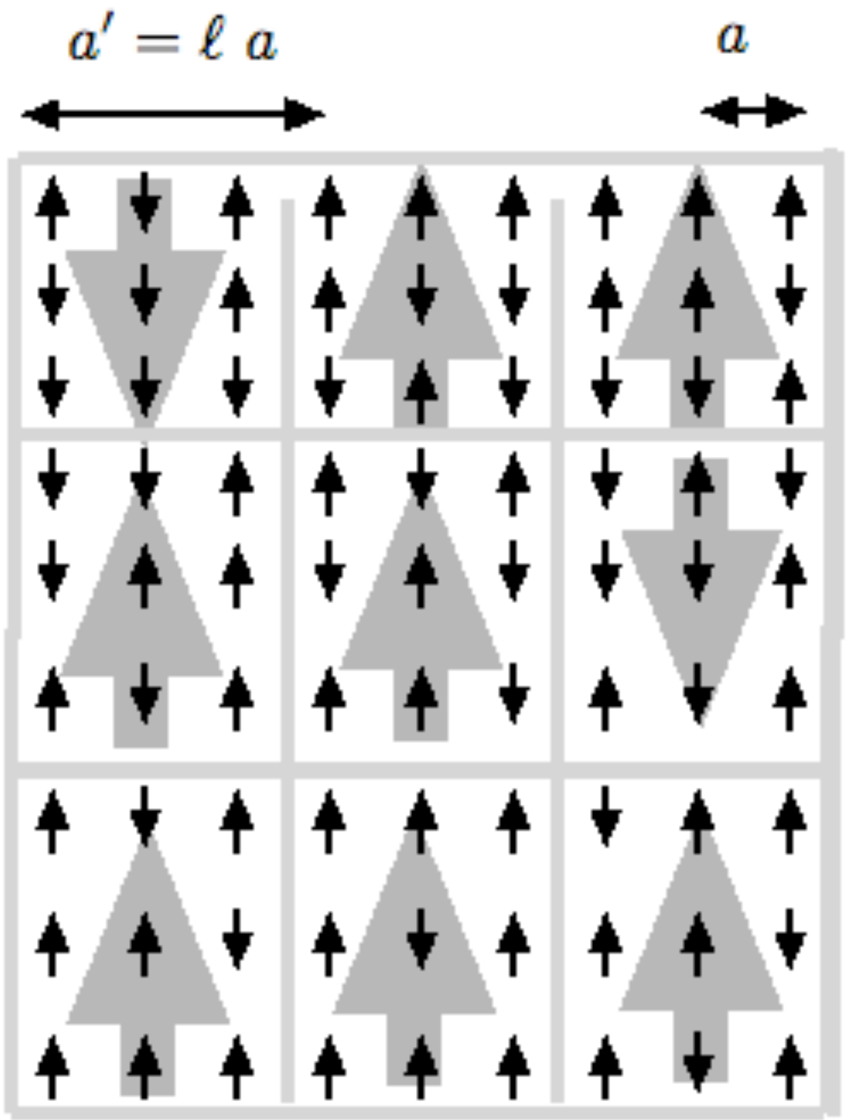
So in 1966 produced a heuristic and incomplete theory of critical behavior

But it does describe scaling

L.P. Kadanoff (1966): "Scaling laws for Ising models near Tc", Physics (Long Island City, N.Y.) 2, 263.

L.P. Kadanoff and W. Gotze and D. Hamblen and R. Hecht and E.A.S. Lewis and V.V. Palciauskas and M. Rayl and J. Swift and D. Aspnes and J.W. Kane Static Phenomena Near Critical Points: Theory and Experiment, Rev. Mod. Phys. **39** 395 (1967).

Now there is a five year pause while the field tries to figure out what to do next



fewer degrees of freedom produces "block renormalization"

Particle Physics RG before Wilson

idea: masses, coupling constants, etc. in Hamiltonian description of problem different from observed masses, coupling constants, etc. . They change with distance scale as particles are "dressed" by effects of the interaction.

RG was then a good phenomenological idea, used in quantum electrodynamics but not really crucial to the particle physics of that day.

Effective values of couplings, charges, etc. also used in statistical physics.

Wilson 1971 produces complete theory

Wilson's changes:

• He consider all possible couplings. So you don't have to guess which couplings to use. The scale change produces a closed algebra of couplings.

• He considers a succession of renormalizations, not just one. So you don't have to guess where a big scale change will take you. You simply follow result of renormalizations.*

• After many renormalizations you eventually reach a fixed point where the couplings stop changing. Each fixed point can be considered to be its own separate physical theory.

K.G. Wilson, Renormalization Group and Critical Phenomena, Phys. Rev. B4 3174-3183 (1971)

* See also earlier work, e.g. Gell-man and Low

Types of Fixed Points

- continue changes in length scale until we reach limits of system (finite system) or
- continue changes in length scale until we reach a situation in which coupling change no more (infinite system)
- The latter is called a fixed point and describes phases

There are three kinds fixed points:

strong coupling: K,h go to infinity describes e.g. liquid phase

weak coupling: K, h go to zero describes e.g. vapor phase

critical: K set to K_c h set to zero, critical point

The different in destinations encode different behavior.

Different symmetries and spatial dimensions produce different fixed points.

Universality

Start from view of microscopic system(s). We want to understand macroscopic behavior near critical point.

1. Adjust relevant couplings so system is near critical
2. Do renormalizations, lots of them, approach macroscopic behavior
3. Notice that irrelevant couplings have renormalized almost to zero. System approaches one of a few distinct fixed points.

Very different starting points reduce to a few distinct fixed points. Different starting systems fall in a few classes called Universality Classes depending upon their eventual fixed point. Each member of universality class has identical critical behavior.

Universality Classes

Ising model universality class:

ferromagnet with easy axes

liquid gas phase transition

XY model universality class:

magnet with easy plane of magnetization

normal fluid to superfluid transition

in (d=2) also solid to liquid transition

Renormalization Group produces big change

old way: start with ensemble (like canonical ensemble) find averages

new way: start with ensemble calculate new ensemble.

after many renormalizations, find fixed point

- at weak coupling fixed point: find averages
- at critical fixed point: find scalings
- at strong coupling fixed point: find theory of nontrivial behavior, e.g. elasticity, acoustics, ferromagnetism, superconductor. Connect theories on different length scales.

Extended Singularity

Each universality class shows a connection between a microscopic internal symmetry (e.g. Ising model's up & down) or (rotation in a plane) and the topological properties of a large hunk of space, much larger than the range of the forces. It shows thermodynamic singularities, correlation functions which fall off algebraically, internal parameters, e.g. coherence length =inverse particle mass that have singular behavior.

This connection between macroscopic and microscopic is interesting and quite beautiful.

Extended Singularity

Each universality class shows a connection between a microscopic internal symmetry (e.g. Ising model's up & down) or (rotation in a plane) and the topological properties of a large hunk of space, much larger than the range of the forces. It shows thermodynamic singularities, correlation functions which fall off algebraically, internal parameters, e.g. coherence length =inverse particle mass that have singular behavior.

This connection between macroscopic and microscopic is interesting and quite beautiful.

So I disagree with Dirac, who said that renormalization is ugly. If you believe in a world of rich physics, and of many different theories, renormalization provides a quite elegant connection among theories.

The Crucial Ideas-for the revolution

Ideas:

- Criticality: recognized as a subject in itself
- Scaling: Behavior has invariance as length scale is changed
- Universality: Expect that critical phenomena problems can be divided into different "universality classes"
- Running Couplings: Depend on scale. Cf. standard model based on effective couplings of Landau & others.
- effective fields of all sorts: Running couplings are but one example of this.
- Fixed Point: Singularities when couplings stop running. K. Wilson
- Renormalization Group: K. Wilson (1971), calculational method based on ideas above.

Each Item, except RG, is a "consensus" product of many minds

The Outcome of Revolution

Excellent quantitative and qualitative understanding of phase transitions in all dimensions. Information about

- **Universality Classes**

All problems divided into “Universality Classes” based upon dimension, symmetry of order parameter, ...

Different Universality Classes have different critical behavior

e.g. Ising model, ferromagnet, liquid-gas are in same class
XYZ model, with a 3-component spin, is in different class

To get properties of a particular universality class you need only solve one, perhaps very simplified, problem in that class.

moral: theorists should study simplified models. They are close to the problems we wish to understand

Conceptual Advances

First order phase transition represent a choice among several available states or phases. This choice is made by the entire thermodynamic system.

Critical phenomena are the vacillations in decision making as the system chooses its phase.

Information is transferred from place to place via local values of the order parameter.

There are natural thermodynamic variables to describe the process. The system is best described using these variable.

Each variable obeys a simple scaling.

Next: After the Revolution:

References

jfi.uchicago.edu\~leop\

More is the Same: Mean Field Theory and Phase Transitions also in *J. of Stat. Phys.*

Theories of Matter: Infinities and Renormalization

Cyril Domb *The Critical Point* Taylor and Francis (1996).

Benjamin Widom, J. Chem. Phys. **43** 3892 and 3896 (1965).

K.G. Wilson, Renormalization Group and Critical Phenomena, Phys. Rev. **B4** 3174-3183 (1971)

LPK, Scaling Laws for Ising Models Near T_c, *Physics* **2** 263 (1966).

J.M.H. Levalt Sengers, *How Fluids Unmix*, Koninklijke Nederlandse Akademie van Wetenschappen, Amsterdam (2002)

L.P. Kadanoff and W. Gotze and D. Hamblen and R. Hecht and E.A.S. Lewis and V.V. Palciauskas and M. Rayl and J. Swift and D. Aspnes and J.W. Kane Static Phenomena Near Critical Points: Theory and Experiment, Rev. Mod. Phys. **39** 395 (1967).

Jan V. Sengers and Joseph G. Shanks. Experimental Critical-Exponent Values for Fluids, J. Stat. Phys. **137** 857-877 (2009).

M. E. Fisher, The theory of equilibrium Critical Phenomena, *Reports on Progress in Physics* **XXX**, **part II** 615 (1967).

P. Heller, Experimental Investigations of Critical Phenomena, *Reports on Progress in Physics* **XXX**, **part II** 731 (1967).

L.D. Landau, Phys. Z. Sow.**11**, 26 545 (1937). English Translation: *Collected Papers of L. D. Landau*, D. ter Haar, ed., pp. 193-215. Pergamon Press, Oxford, (1965).

P. W. Anderson, More is different, *Science* **177**, 393-396 (1972)

More references

M. Smoluchowski, Ann. Phys. Leipzig **25**, 205 (1912).

L.S. Ornstein and F. Zernike, Proc, Acad Amsterdam **17** 793 (1914).

- E.C.G. Stueckelberg, A. Peterman (1953): Helv. Phys. Acta, 26, 499.
- Murray Gell-Mann, F.E. Low (1954): Phys. Rev. 95, 5, 1300. The origin of renormalization group
- N.N. Bogoliubov, D.V. Shirkov (1959): The Theory of Quantized Fields. New York, Interscience. The first text-book on the renormalization group method.
- L.P. Kadanoff (1966): "Scaling laws for Ising models near Tc", Physics (Long Island City, N.Y.) 2, 263. The new blocking picture.
- C.G. Callan (1970): Phys. Rev. D 2, 1541.[2] K. Symanzik (1970): Comm. Math. Phys. 18, 227.[3] The new view on momentum-space RG.
- K.G. Wilson (1975): The renormalization group: critical phenomena and the Kondo problem, Rev. Mod. Phys. 47, 4, 773.[4] The main success of the new picture.
- S.R. White (1992): Density matrix formulation for quantum renormalization groups, Phys. Rev. Lett. 69, 2863. The most successful variational RG method.