Anomalous Electrical Conductivity above the Superconducting Transition*

LEO P. KADANOFF AND GEORGE LARAMORE†

Department of Physics, University of Illinois, Urbana, Illinois
(Received 2 July 1968)

The rise in the electrical conductivity as a material approaches its superconducting transition is described in terms of fluctuation theory. About $T_c$, small regions become superconducting and can carry an anomalous current, thereby causing a rise in conductivity. The use of the Kubo formula permits the estimation of the contribution from this source. A comparison of this estimate with previous theories and experiment is carried out.

INTRODUCTION

In two recent experiments involving highly resistive thin films just above their superconducting transition, an anomaly was observed in the electrical conductivity of the form

$$\frac{\sigma - \sigma_m}{\sigma_m} = A \left[ T_c/(T - T_c) \right]^{\gamma},$$

(1)

where $\sigma_m$ is the conductivity far above $T_c$. Relatively far from the transition, the critical index $x$ was observed to be close to unity, while nearer the transition, the “constants” $x$ and $A$ changed, with $x$ apparently becoming smaller than 1. This paper is devoted to discussing these observations in terms of time-dependent critical fluctuation theory.

At first sight, the result $x=1$ is very puzzling. It is well known that over a large range the superconducting transition is described by a variant of mean field theory—the Ginzburg-Landau theory. In fact, the standard fluctuation theory is built upon the Ginzburg-Landau free energy. But, it is also well known that in the mean field theories, critical fluctuations give a square-root divergence in transport coefficients that is $x=\frac{1}{2}$. This general point is illustrated by a specifically superconducting calculation of Maki, who finds $x=\frac{1}{2}$ relatively far above $T_c$. Then why does experiment give $x=1$? The reason for this was pointed out to us in Ref. 2, in the work of Abrahams and Woo, and also by Hohenberg.

Near $T_c$, the films under study in Refs. 1 and 2 are thin compared with the coherence distance $\xi$ which describes the length scale for coherent fluctuations. (In the mean field theory, the relevant length is the Ginzburg-Landau coherence length.) Consequently, while the bulk system has a volume of a typical fluctuation,

$$\Omega_{\text{eff}} \sim \xi^2 (\text{bulk}),$$

(2a)

thin a film of width $L$ has an effective volume for fluctuations

$$\Omega_{\text{eff}} \sim \xi^2 L (\text{film}),$$

(2b)

while a wire, which is thin compared to the coherence length, will have a typical fluctuating volume proportional to its cross-sectional area $S$, i.e.,

$$\Omega_{\text{eff}} \sim \xi S (\text{wire}).$$

(2c)

These fluctuating volumes help determine the anomalous conductivity. According to the Kubo formula the conductivity is proportional to the mean-squared fluctuations in the current, and, as we shall see, this mean-squared fluctuation is inversely proportional to the fluctuating volume. Hence, in the films the right-hand side of Eq. (1) is enhanced by an extra factor of $\xi/L$ relative to its value in the bulk. Since, in the Ginzburg-Landau theory, the coherence length at temperature $T$ near $T_c$ is enhanced by a factor $[(T-T_c)/T_c]^{1/3}$ relative to its zero-temperature value,

$$\xi = \xi_{0}\sim (T_{0}/T)^{1/3} \left[ T_{c}/(T - T_c) \right]^{1/3},$$

(3)

the anomalous term is enhanced by an extra factor

$$\left[(T_{0}/T)^{1/3}L\right]\left[ T_{c}/(T - T_c) \right]^{1/3}$$

in the film relative to the bulk. Hence $x=\frac{1}{2}$ in the bulk implies $x=1$ in the film and $x=\frac{3}{2}$ in the wire. This thin-film effect is described in Ref. 7, which derives the result $x=1$ and $\sigma_{\text{eff}}A = c^2/16 \kappa L$.

DERIVATION OF ANOMALOUS CONDUCTIVITY

As discussed in Ref. 2, the basic fluctuations that can produce the anomalous electrical conductivity are in temperature and the Ginzburg-Landau complex-order parameter $\psi$. Basically, the anomalous conduc-

* This research was supported in part by the National Science Foundation under Grant NSF GP 7765.
† National Science Foundation predoctoral fellow.
1 R. E. Glover, Phys. Letters 25A, 542 (1967). We would like to thank R. E. Glover and R. A. Ferrell for making the numerical data of this experiment available to us.
9 P. C. Hohenberg (private communication).
tivity is produced via a time-dependent fluctuation in temperature that pushes a region of the system below \( T_o \) and a simultaneous fluctuation in \( \psi \) (or \( \nu_i \)) that allows a large current flow. Mathematically, a temperature fluctuation of wave vector \( q \) relaxes according to the Fourier heat-conduction equation with a relaxation rate

\[
s_\tau(q) = \frac{1}{\tau} \frac{\lambda}{\rho C_p} q^2 ,
\]

(4a)

\( \lambda \) being a thermal conductivity. To find the relaxation rate for fluctuations in \( \psi \) (or \( \nu_i \)) one can solve the \( T \) equation\textsuperscript{12–18} in the presence of impurities. This gives a relaxation rate above \( T_c \) of the form

\[
s_\phi(q) = a^2 \frac{b(T-T_c)}{h} + q^2 (\nu^2/3) \tau.
\]

Here \( a \) is a constant of order unity, \( \nu_T \) is the Fermi velocity, and \( \tau \) is the mean free time. Physically, this relaxation can be connected with disappearance (first term) and diffusion (second term) of excitations involving flow of “supercurrent” above \( T_c \). If one accepts the Einstein relation between conductivity and diffusivity, one can also write this decay rate as

\[
s_\phi(q) = a^2 \frac{b(T-T_c)}{h} + \frac{\sigma}{\epsilon \tilde{\epsilon}(\tilde{\epsilon}/\mu \tilde{\tau})} q^2 .
\]

(4b)

Since the diffusion term in (4b) is not tied to the weak coupling limit of the \( T \) approximation, we can think of this term as a general result that is valid even when the fluctuations are large. The term independent of \( q \) is, however, a result of perturbation theory and hence much less generally valid.

The modes in question can be represented by creation operators \( \alpha_T(q) \) and \( \alpha_\phi(q) \) that have the effect of adding a disturbance with wave vector \( q \) to a typical equilibrium state \( \vert \rangle \). Since the thermal mode involves entropy fluctuations and the \( \phi \) mode involves current flow, we expect\textsuperscript{13} \( \alpha_T(q) \sim s_\phi(q) \), which is the entropy operator, and \( \alpha_\phi(q) \sim J_\phi(q) \), where \( J_\phi \) is the supercurrent operator. Properly normalized versions of these operators are

\[
\alpha_T(q) = s_\phi(q)/[ \langle \vert s_\phi(q)s_\phi(-q) \vert \rangle]^{1/2},
\]

\[
\alpha_\phi(q) = J_\phi(q)/[ \langle \vert J_\phi(q)J_\phi(-q) \vert \rangle]^{1/2}.
\]

(5)

The actual calculation of the conductivity is based upon the formalism of Ref. 14. The Kubo formula for the conductivity is converted into a sum over intermediate fluctuation states of the form

\[
\sigma = (k_B T)^{-1} (\Omega)^{-1} \sum_\alpha \langle \langle J \vert \alpha \rangle \vert \alpha \rangle ^2 /s_\alpha ,
\]

(6)

where \( s_\alpha \) is the relaxation rate of the state \( \vert \alpha \rangle \) and \( \Omega \) is the volume of the system. Most of the possible intermediate states contribute to the temperature-independent term \( s_\alpha \). We look specifically for the anomalous

term produced by the intermediate states

\[
\vert q \rangle = a_\phi(-q) a_T(q) \rangle ,
\]

which involves the two modes discussed above. These intermediate states give a contribution to the conductivity

\[
\sigma - \sigma_\phi = (\Omega)^{-1} \sum_\alpha \langle \langle M_\phi \rangle^2 / \vert \langle J_\phi(q) + s_\phi(-q) \rangle \rangle ,
\]

with

\[
\langle M_\phi \rangle^2 = (k_B T)^{-1} \left\langle \left| \langle J_\phi(-q)J_\phi(q) \rangle \right| ^2 \right\rangle
\]

\[
= (k_B T)^{-1} \left[ \left| \langle J_\phi(-q)J_\phi(q) \rangle \right| ^2 \right] /
\]

\[
\left[ \left| \langle J_\phi(-q)J_\phi(q) \rangle \right| ^2 \right].
\]

(7)

As in Ref. 14, the relaxation rate of the composite mode has been approximated in Eq. (6) by the sums of the relaxation rates of the individual modes.

Now we turn to estimating the various quantities on the right-hand side of Eq. (6). The main contribution to the sum over wave vectors is expected to arise when \( q \ll \) the coherence length. Hence, in the bulk when \( q \ll \) the inverse coherence length. Hence, in the bulk system we can replace \( \Omega^{-1} \sum_\alpha \langle \vert J_\phi(q) \rangle \rangle with \( \langle \vert J_\phi(q) \rangle \rangle \). (2)

The same argument works in dealing with the film. Wave vectors parallel to the film surface are integration variables. The sum over wave vectors perpendicular to the surface contains only one contribution when \( L \ll \). In the end, we find that Eqs. (8) and (2) apply for the film and wire geometries as well.

At the relevant wave vector \( q \sim \xi^{-1} \), the frequency denominator is

\[
s_\phi = s_\phi(q) + s_T \sim \xi^{-2} \left[ \frac{\lambda}{\rho C_p} + \frac{\sigma}{\epsilon \tilde{\epsilon}(\tilde{\epsilon}/\mu \tilde{\tau})} \right] .
\]

(9)

In the estimate (9) of \( s_\phi \) we drop the first term in Eq. (4b) because, when \( q \sim \xi^{-1} \), the scaling idea\textsuperscript{19} suggests that the two terms should be of the same order of magnitude. Hence we use the second (and better-known) term and assume that the first term is also of this same size. These terms are indeed of the same size whenever the mean field theory is valid, but we assume that this identity always holds near \( T_o \).

The final step in evaluating the conductivity is the estimation of the matrix elements on the right-hand side of Eq. (7). For \( q \sim 0 \) and \( T \ll T_o \), the operators can


\textsuperscript{13} The \( T \) equation is used in Refs. 6, 7, and 9 to calculate \( \sigma - \sigma_\phi \).

\textsuperscript{14} L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968).

be replaced by thermodynamic derivatives\(^8\) so that the right-hand side of Eq. (7) can be evaluated exactly as
\[
|\bar{q}|^2 = k_B T_0 \sum_{\xi \neq 0} e^{\xi^2} (\langle \partial^2 T / \partial^2 T \rangle_{(T_0 \ldots T_0)}^{\xi^2})^2 \frac{\partial^2 C_p}{\partial T^2} \rho C_p m^2 \rho_0.
\]

Near \( T_0 \), Eq. (10) may be simplified by writing \( \rho T_0 / \partial T \sim \rho T_0 / (T - T_0) \) and using a relation, defined by Josephson,\(^7\) that is correct in both the scaling\(^8\) and mean field (Ginzburg-Landau) theories:
\[
\frac{1}{2} p (h/m^2)^{\xi^2} - \rho C_p m^2 \left( \frac{T - T_0}{T_c} \right)^{\xi^2},
\]
where \( C_p \) is the electronic heat capacity. Hence, Eq. (10) simplifies to
\[
|\bar{q}|^2 \sim |k_B T / h^2| \sum_{\xi \neq 0} e^{\xi^2} \left( \frac{\partial^2 C_p}{\partial T^2} \right) \frac{\rho C_p m^2 \rho_0}{\rho C_p m^2 \rho_0} \quad \text{for} \quad q \lesssim \xi^{-1}.
\]

In both scaling\(^8\) and mean field theories\(^8\) there is a symmetry in the order of magnitude of both \( \xi \) and of fluctuations under the interchange \( T \leftrightarrow T_0 \). Hence, we shall assume that Eq. (12) holds for all values of \( T \) near \( T_0 \).

Notice the divergence in Eq. (11) as \( T \to T_0 \) so that \( \xi \to \infty \). This very large matrix element only appears for small \( q \). As \( g \) gets larger than \( \xi^{-1} \), the matrix element gets smaller and the sum over vector modes is cut off.

At this point, we substitute Eqs. (8), (9), and (12) in Eq. (6) and find
\[
\sigma = \sigma_0 e^{\xi^2} \frac{k_B T}{\Omega_{\text{eff}}} \left( \frac{\lambda}{\rho C_p} \right) \left( \frac{\sigma}{\rho C_p} \right) \left( \frac{\partial^2 C_p}{\partial T^2} \right) \frac{\rho C_p m^2 \rho_0}{\rho C_p m^2 \rho_0}.
\]

Equation (13) is our basic result. We now apply this result to a variety of physical situations. The effective volumes given by Eq. (2) are temperature-dependent because \( \xi_0 L \) is temperature-dependent. Since we wish to isolate the dependence on \( T - T_0 \) we replace the parameter \( \Omega_{\text{eff}} \) by the parameter \( \Omega_{\text{eff}} \), which is the zero-temperature fluctuation volume,
\[
\Omega_{\text{eff}} = \left( \frac{\xi_0 L}{T_0} \right)^{\frac{3}{2}}
\]
\[
= \xi_0 L
\]
\[
= \xi_0 L^{1/2} S,
\]
to write Eq. (13) as
\[
\sigma = \sigma_0 e^{\xi^2} \frac{k_B T}{\Omega_{\text{eff}}} \left( \frac{\lambda}{\rho C_p} \right) \left( \frac{\sigma}{\rho C_p} \right) \left( \frac{\partial^2 C_p}{\partial T^2} \right) \frac{\rho C_p m^2 \rho_0}{\rho C_p m^2 \rho_0}.
\]

Relatively far from \( T_0 \), \( \sigma \approx \sigma_0 \). Then \( A \) is a temperature-independent constant. In this case, we write \( A \) as \( A_\infty \) and the entire temperature dependence of the expression is in the \( [T_0 / (T - T_0)]^{\xi^2} \) term. The theory of Abrahams and Woon\(^3\) gives results quite similar to this. In particular, for the film we see a linearly divergent anomalous term as indicated by experiment\(^1,2\) and other previous theory.\(^7\) However, very near \( T_0 \) we should expect to see a new temperature dependence. There are two sources of this change. One is that the conductivity in the denominator of \( A \) begins to show an appreciable temperature dependence caused by the anomalous term in \( \sigma \). The other is that \( \xi \) becomes different from the Ginzburg-Landau coherence length so that Eq. (15) fails. Before discussing the second effect, we compare Eq. (15) with experiment.

According to the mean field result, Eq. (15), very near \( T_0 \), the conditions
\[
\sigma > \sigma_0 \quad \text{(16a)}
\]
\[
\sigma > \sigma_0^e (\partial^2 / \partial T^2) \quad \text{(16b)}
\]
are satisfied so that the term proportional to anomalous conductivity should dominate the behavior of the denominator of \( A \). In this region very close to \( T_0 \),
\[
\frac{\sigma}{\sigma_0} = B \left( \frac{T_0}{(T - T_0)} \right)^{\xi^2},
\]
\[
B = \frac{e^{\xi^2}}{\xi_0 L} \left( \frac{k_B T}{\Omega_{\text{eff}}} \right)^{\frac{3}{2}}.
\]

Here \( x \) has the values listed in Eq. (15).

The behavior predicted by Eq. (15) is qualitatively similar to the observed anomalies.\(^1,2\) To make a quantitative comparison, it is necessary to evaluate the constant \( A \). Assume that the electronic contribution to \( \lambda / \rho C_p \) and \( C_p \) dominates.\(^9\) Then the free-electron theory can be used to obtain an estimate of \( A \) in the region where \( \sigma \approx \sigma_0 \) and we obtain
\[
A_\infty \sim a_0 \xi_0^2 \Omega_{\text{eff}},
\]
where \( a_0 \) is the typical distance between conduction electrons, \( a_0 \sim h / (m v_0) \). Before we consider a film in some detail, we shall first use Eq. (18) to obtain an estimate of the width of the transition region for a bulk system and for a wire.

In a dirty bulk system we expect
\[
\frac{\sigma - \sigma_0}{\sigma_0} = (a_0^2 / \xi_0^2) \sigma_0 \frac{\rho C_p}{\rho C_p} e^{\xi^2},
\]
where \( \rho C_p / \rho C_p \) is the transition width can be defined as the size of \( \varepsilon \) when \( \sigma = \sigma_0 \). At this point, \( \varepsilon = a_0^2 / \xi_0^2 \).

\[\text{(19)}\]
\[\varepsilon = a_0^2 / \xi_0^2.\]

Relatively far from \( T_0 \), the relaxation rate \( [\sigma / \rho C_p (\partial^2 / \partial T^2)]^{-1} \) is sufficiently rapid so that the lattice cannot follow the temperature fluctuations in the electronic system. Hence, far from \( T_0 \), in Eq. (13), \( \lambda \) and \( C_p \) should be replaced by their electronic components. On the other hand, closer to \( T_0 \), the relaxation rate is slower, the lattice can follow, and then the estimate given by Eq. (13) should include lattice effects in \( \lambda \) and \( C_p \). This point is discussed in more detail below.
For amorphous bismuth where $a_0 \sim 0.61$ Å, $\xi \sim 3600$ Å, and $l \sim 5$ Å, we have a transition width $\epsilon \sim 3 \times 10^{-2}$.

For a wire,

$$\frac{(\sigma - \sigma_\infty)}{\sigma_\infty} \sim (a_0^2 \xi^2 / l^2) S e^{-l^2} \tag{21}$$

so that $\sigma = 2\sigma_\infty$ implies

$$\epsilon \sim (a_0^2 \xi^2 / l^2) S l^2 \tag{22}$$

When the wire diameter is small compared to the electronic mean free path in the bulk material, surface scattering limits the effective electronic mean free path to the diameter of the wire.\(^{20}\) For the case of a tin whisker of cross-sectional area $S = 10^{-4}$ cm\(^2\) we find a width $\epsilon \sim 2 \times 10^{-9}$.

From a thin dirty film,

$$A_\infty \sim a_0^2 / IL. \tag{23}$$

This should be compared with the exact calculation of Aslamazov and Larkin\(^{2}\) for the case in which lattice effects are ignored. They find

$$A_\infty = 1.85 (a_0^2 / IL). \tag{24}$$

Except for the numerical factor our results agree, and this is quite reasonable since we neglected numerical factors in our estimate of the sum over matrix elements in Eq. (6). We shall use Eq. (24) for our subsequent comparison with experiment. For the Glover\(^{2}\) experiment on an amorphous bismuth film having a background resistance of 31.13 Ω/□ we find $A_\infty \sim 4.4 \times 10^{-4}$ in quite good agreement with the experimental result of $A_\infty = 4.7 \times 10^{-4}$. Glover\(^{21}\) has recently sent us information on another bismuth film. This film had a background resistance of 28.1 Ω/□ and an experimental value of $A_\infty = 4.1 \times 10^{-4}$. From Eq. (24) we find $A_\infty \sim 4.2 \times 10^{-4}$, which is again in quite good agreement with experiment.

Shier and Ginsburg\(^{22}\) have measured the transition in bismuth films that were on the order of 1000 Å thick. Equation (24) predicts a transition width of about $10^{-3}$ K, which is in fair agreement with the experimental value of $(5 \times 10^{-3})$ K. They also analyze the contribution to $\sigma - \sigma_\infty$ due to fluctuations in the bulk and obtain results for the width of the transition region similar to our own. They argue that relatively far from $T_c$ fluctuations become exponentially unlikely because the Boltzmann factor $e^{-E/kT}$ prohibits fluctuations with a very large free energy. On the other hand, we argue that above $T_c$ small amplitude fluctuations can appear with arbitrarily small free energies, the free-energy difference being proportional to the squared amplitude of the fluctuations. Hence, we obtain a different $T - T_c$ dependence of $\sigma - \sigma_\infty$.

In the experiment of Ref. 2 the appearance of the superconductor as inclusions in an oxide matrix might be expected to make the phonon contribution to the thermal conductivity become more significant. However, using Eq. (24) with $l \sim 100$ Å and $a_0 \sim 1.8$ Å we find values of $A_\infty \sim 1.1 \times 10^{-2}$ for the $l = 5$ Å film and $\sim 3.0 \times 10^{-3}$ for the $l = 0.2$ Å film. The respective experimental values for these films are $1.1 \times 10^{-2}$ and $10^{-3}$. It is possible that part of the disagreement in the $l = 0.2$ Å film can be attributed to the thicker oxide layer between the aluminum particles making lattice effects more important.

In order to further test the agreement between theory and experiment, Eq. (13) was rewritten for the case of the film as

$$\frac{\sigma - \sigma_\infty}{\sigma_\infty} = \left( \frac{(D+1) A_\infty T_c / (T - T_c)}{D + \sigma / \sigma_\infty} \right) \tag{25}$$

where $D$ is the ratio of the thermal diffusivity to the electrical diffusivity corresponding to the conductivity $\sigma_\infty$. For only an electronic contribution to the thermal diffusivity we expect $D \sim 1$.

A conventional comparison between theory and experiment is made in Fig. 1 for the bismuth film of Glover\(^{2}\) The agreement is reasonable down to $R/R_\infty \sim 0.5$. In making this fit we attempted to fit as much of the data as possible with the form given by Eq. (25). This

\[^{21}\text{R. E. Glover (private communication).}\]
\[^{22}\text{J. S. Shier and D. M. Ginsburg, Phys. Rev. 147, 384 (1966).}\]
fit is somewhat deceptive; for if
\[
\left[ \frac{\sigma - \sigma_{\infty}}{\sigma_{\infty}} \right] (D + \sigma/\sigma_{\infty})/(D+1) \]
is plotted versus \( T \), our theory predicts a straight line from which \( T_\lambda \) could be determined. However, as can be seen in Figs. 2 and 3, the fit to a straight line is not good at values of \( \sigma \geq 2\sigma_{\infty} \). It should also be noted that \( D \) is a relatively insensitive parameter with the data being fit about equally well over a wide range of values of \( D \) and a different \( T_\lambda \) resulting from each value. An interesting thing about the plots is the apparent break at about \( T = 6.054^\circ \text{K} \). There are two possible explanations for this:

(a) Note that the parameters \( A_{\infty} \) and \( D \) really contain temperature-dependent terms such as \( C_\tau \) and \( \lambda \). They are the actual specific heat and thermal conductivity of the system. Although any temperature variation due to these quantities might appear quite small, it is possible for large variations to occur because of a change in the coupling between the electrons and the lattice. Relatively far from \( T_\lambda \), the fluctuations are of a short-wavelength–high-frequency nature, and the lattice cannot respond to the fluctuations in the electronic temperature. It is in this region that we expect (and obtain) good agreement with the estimates of \( A_{\infty} \) and \( D \) given by “free-electron” theory. But near \( T_\lambda \) the fluctuations are of a long-wavelength–low-frequency type, and the lattice temperature can couple to the electronic temperature fluctuations. This would have

the effect of changing the specific heat in \( A_{\infty} \) to \( C_\tau^g + C_\tau^e \) and changing the thermal diffusivity to

\[
\frac{(\lambda^g + \lambda^e)}{\rho (C_\tau^g + C_\tau^e)},
\]
where the superscript \( g \) denotes the lattice contribution. This would change both \( A_{\infty} \) and \( D \) and could qualitatively account for the break in the curve.

(b) Another possible cause of this break is the breakdown of the mean field theory (MFT) which was used to evaluate \( \xi \) and \( \Omega_{\text{eff}} \) in Eq. (13). The Ginzburg13 criterion for the breakdown of MFT and the advent of the critical region is given by

\[
\Delta F \geq \frac{1}{2} \rho C_\tau \Omega_{\text{eff}}(\xi)^2 \sim k_B T,
\]
where \( \Delta F \) is the free energy associated with a typical fluctuation. We thus expect MFT to break down when

\[
e^2 \sim \frac{1}{2} \rho C_\tau \Omega_{\text{eff}} \sim k_B T.
\]

Let us compare this with the result of Eq. (13) when \( \sigma \sim 2\sigma_{\infty} \). Using free-electron theory to estimate the parameters involved and using the Ginzburg-Landau coherence length for \( \xi \), we find that

\[
e^2 \sim \frac{1}{2} \rho C_\tau \Omega_{\text{eff}} \sim k_B T.
\]

Equations (27) and (28) are essentially equivalent

---

and so for $\sigma \geq 2a_0$ we can no longer use MFT quantities to evaluate Eq. (13).

CRITICAL FLUCTUATIONS

To see how critical fluctuations can modify the conductivity results, let us return to the bulk case. In that case, we should expect molecular field theory, i.e., Eq. (15) to hold until $(\sigma - \sigma_0) \sigma_0 \sim 1$. If the lattice terms do not swamp the electronic terms in $\lambda / \rho C_p$ then this condition for noncriticality derived from Eq. (15) is equivalent to the result of Ref. 23, i.e.,

$$\frac{T - T_c}{T_c} \approx \frac{\sigma}{\sigma_0} \approx \frac{\tau}{\tau_0} = \epsilon_v. \quad (29)$$

For smaller values of $(T - T_c) / T_c$, $\sigma / \sigma_0 \gg 1$. Also we expect that the coherence length should have the usual temperature dependence for the critical region, i.e., $\xi \sim T^{-2/3}$. Then Eq. (13) implies that

$$\sigma \sim T^x / \xi_0 \sim T^{-2/3} \xi^{-2/3} \sim T^{-2/3}$$

so that, when we demand that the conductivity vary continuously at $(T - T_c) / T_c = \epsilon_v$, we find

$$\sigma / \sigma_0 \sim (\epsilon / \epsilon_v)^{2/3} \quad \text{for} \quad \epsilon_v > (T - T_c) / T_c > 0. \quad (30)$$

This result was first found by Maki. It is interesting to notice that the analogous quantity in the superfluid transition, the thermal conductivity, is believed to diverge in the same way.

But, what of the films? According to the work of Rice and Hohenberg, the superconducting transition cannot really take place in the usual way in a film. In fact, according to Langer, $\sigma$ never really goes to infinity in a film, it just becomes very large. One way of expressing this result is to say that even below the nominal transition temperature, the range of coherent order, $\xi$, becomes larger and larger but never goes to infinity.

Assume that this is what indeed occurs and Eq. (13) remains valid even below the nominal $T_c$. Then, when applied to the high-conductivity film, Eq. (13) implies

$$\sigma / \sigma_0 \sim \left(\frac{\hbar}{\hbar_0} \right)^{1/2} \frac{k_B T}{L} \frac{\xi}{\hbar} \quad (31)$$

so that $\sigma$ appears to grow as $T$ drops further and further below $T_c$ at the same rate as the coherence length.

According to the mean-field-theory calculation of Rice, when $T < T_c$, the order-parameter correlation function drops off in space as

$$\langle \psi(R) \psi^*(0) \rangle \sim \exp \left[ - \frac{1}{\pi L} \frac{m}{\rho \epsilon_v} (\hbar)^2 \ln \frac{R}{\xi_{0L}} \right]. \quad (32)$$

where $\xi_{0L}$ is the mean-field-theory coherence length. Equation (32) holds for $R \geq \xi_{0L}$. Now identify the actual coherence length as the distance over which the correlation function drops to $1/e$ of its value at $R = \xi_{0L}$. Then we conclude that

$$\ln(\xi / \xi_{0L}) \sim \frac{\pi L \rho_0}{k_B T} \frac{\hbar}{m} \Rightarrow \xi = \xi_{0L} \exp \left[ \frac{\pi L \rho_0}{k_B T} \frac{\hbar}{m} \right].$$

This result for the $T < T_c$ coherence length can be equivalently written as

$$\xi \sim \xi_{0L} \exp(-\epsilon / \epsilon_v), \quad (33)$$

where $-\epsilon$ is the positive quantity $-(T - T_c) / T_c$ and $\epsilon_v$ is a measure of the value of $\epsilon$ for which an exponential increase in coherence length occurs,

$$\epsilon_v = \left[ k_B (T_c - T) / \pi L \rho_0 \right] (m / \hbar) \sim A_{\omega} \omega. \quad (34)$$

In the mean field theory $\epsilon_v$ is temperature-independent near $T_c$. In fact $\epsilon_v$ is, except for a numerical factor, the same as $A_{\omega}$ defined by Eq. (24). Above $T_c$, we found that for

$$\epsilon >> \epsilon_v \quad \text{implies} \quad (\sigma - \sigma_0) / \sigma_0 \ll 1. \quad (35a)$$

Now below $T_c$ we can say

$$-\epsilon >> \epsilon_v \quad \text{implies} \quad \sigma / \sigma_0 \gg 1, \quad (35b)$$

since Eq. (31) can be equivalently expressed as

$$\sigma / \sigma_0 \sim (\epsilon / \epsilon_v)^{1/3} \exp(-\epsilon / \epsilon_v). \quad (36)$$

Hence, $A_{\omega}$ or equivalently $\epsilon_v$ are measures of the width of the intrinsic resistive transition that occurs in the film.

Equation (36) represents a guess about the behavior of the conductivity well below the nominal superconducting transition temperature. The reader will recognize the many pitfalls and weaknesses in the argument that led up to the result. A real understanding of the conductivity in this region requires more careful theoretical investigation. This understanding would also be helped by a very careful experimental study of the region just below the nominal superconducting transition.

---

24 See the work of Ferrell et al. (Ref. 15).
28 J. S. Langer (private communication).