Dimensional Calculations for Julia Sets

This content has been downloaded from IOPscience. Please scroll down to see the full text.

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 128.135.12.127
This content was downloaded on 26/02/2015 at 15:59

Please note that terms and conditions apply.
Abstract
Julia sets are strange repellers which arise from mapping problems involving analytic functions of complex variables. Those sets are multifractals and have a kind of scale invariance. Relationships between various methods of calculating the scaling properties of these sets are discussed.

1. Introduction
For many dynamical systems there are two apparently different ways of obtaining dimensional data on strange sets: one using the distances between points in the set [1] and the other using derivatives of the mapping function [2]. In this note I examine these two methods for the case of the Julia set of the complex transform:

\[ g(z) = z^2 + c \]  (1.1)

and show that these two approaches give the same answer, at least in the simple limits of small \( c \) and large \( c \).

The calculation is based upon the application of ideas of statistical mechanics to the strange set. The particular approach of references 2 depends upon the descriptions of the set in terms of symbol sequences, i.e., a vector of infinite length [3]

\[ \Sigma = e_0, e_1, e_2, e_3, \ldots \]

with

\[ e_j = \pm 1 \]

used to describe the Julia set. In particular, one sees that every element of the Julia Set may be written as \( z(\Sigma) \), where the significance of this form of writing is that the earlier elements in the symbol sequence play the largest role in determining \( z(\Sigma) \). Specifically, we shall assume that if \( \Sigma \) and \( \Sigma' \) have their first \( n \) digits identical then

\[ |z(\Sigma) - z(\Sigma')| < A 2^{-\alpha n} \]  (1.2)

where \( A \) and \( \alpha \) are constants, which do depend upon \( c \), each of which is greater than zero.

The other major element of the definition of the symbol sequence is that the mapping (which is 2 to 1) has the effect of dropping the first element of the symbol sequence, namely

\[ g(z(\Sigma)) = z(S(\Sigma)) \]

where

\[ S(\Sigma) = (e_0, e_1, e_2, e_3, \ldots) \rightarrow (e_1, e_2, e_3, \ldots) \]  (1.3)

This paper should be considered to be a kind of mathematical appendix to Ref. [4] in which Jensen, Kadanoff, and Procaccia consider the scaling behavior of Julia sets in the special case in which \( c \) is real and is sufficiently small so that the Julia set is a simple closed curve. This set of conditions will hold when \( -0.75 < c < 0.25 \). Under these circumstances, eq. (1.2) is indeed true so that the considerations of the present paper are valid. Thus, the conclusions of the present paper are applicable to Ref. [4].

The main result is very simple indeed. Following Ruelle [2], define

\[ \Gamma_n(c) = \langle |dg(z(\Sigma))|^{1/n} \rangle_{\Sigma \in \text{Fix}_c} \]  (1.4)

where \( \langle \ldots \rangle \) represents an average over a particular set of \( \Sigma \)-values, i.e., those sequences which belong to \( \text{Fix}_c \), and \( \text{Fix}_c \) is the set of symbol sequences which leads to all the distinct, unstable, periodic points (of period \( n \)) of \( g \). In equation (1.4), \( dg(\Sigma) \) is the derivative of the map composed of \( n \)-times, i.e.,

\[ dg^n(\Sigma) = \prod_{j=1}^{n} dg(z(S^{(j-1)}(\Sigma))) \]

and \( dg(\Sigma) \) is simply \( 2z \). Define also the more general quantity

\[ \Gamma_n(c,\text{SET}) = \langle |dg^n(\Sigma)| \rangle_{\Sigma \in \text{SET}} \]  (1.5)

Finally define yet another type of partition function

\[ Z_n(c,\text{SET}) = \langle |\Delta_n(\Sigma)|^{-1} \rangle_{\Sigma \in \text{SET}} \]  (1.6)

Here \( \Delta_n(\Sigma) \) is a difference between \( z(\Sigma) \) an \( z(C_\Sigma) \), where \( \Sigma' = C_{\Sigma} \) has all digits of \( \Sigma' \) equal to those of \( \Sigma \) except for the \( n \)th, \( e_{n-1} = -e_{n-1} \). Then write

\[ \Delta_n(\Sigma) = z(\Sigma) - z(C_\Sigma) \]  (1.7)

Recall that our basic dimensional data is defined by a quantity \( q(c) \) which can be calculated from \( \Gamma_n(c) \) as

\[ q(c) = 1 = \frac{1}{\ln 2} \lim_{n \rightarrow \infty} \frac{\ln \Gamma_n(c)}{n} \]  (1.8)

The basic result is that we can, within broad limits, define a sequence of sets \( \text{SET}_c \), such that the kind of limit defined in (1.9) exists for \( \Gamma_n(c,\text{SET}_c) \) and \( Z_n(c,\text{SET}_c) \) and such that the limit is independent of the choice. In particular, choose \( \text{SET}_c \) to include all of the first \( n - k \) digits of the symbol sequence \( (e_0, e_1, e_2, \ldots, e_{n-k-1}) \) (for some fixed \( n \)-independent value of \( k \)) with exactly the probability distribution as their appearance in \( \text{SET}_c \). Under those circumstances,

\[ \lim_{n \rightarrow \infty} \frac{\ln \Gamma_n(c,\text{SET}_c)}{n} = \lim_{n \rightarrow \infty} \frac{\ln \Gamma_n(c,\text{SET}_c)}{n} = \ln 2 [q(c) - 1] \]  (1.10)

This major result will be proven below. It implies that there are many many different ways of calculating \( q(c) \) which are equally valid as \( n \rightarrow \infty \).

However, one should recognize that for the purposes of numerical calculation, these methods are not equally good.
\[ \Gamma_n(\tau) \rightarrow N(c)2^{(\alpha_1-1)}[1 + O(e^{-\gamma})] \]

(1.11)

where \( \gamma \) is a \( c \)-dependent constant which is greater than zero. The prefactor, \( N(c) \), is an integer (!) which is unity both for small values of \( c \) and large ones. Hence the convergence rate in (1.9) is exponential in \( n \) while from (1.10) and (1.11) the convergence is, in general, no better than algebraic.

### 2. Partition functions

#### 2.1. Relations among different definitions

To establish the relationship between the two kinds of partition functions (1.6) and (1.7), consider the ratio of the two quantities defining these functions, i.e.,

\[ B_n(\Sigma) = \frac{dg(\Sigma)}{\Delta_n(\Sigma)} \]  

(2.1)

We want to show that this quantity is uniformly bounded, i.e., that for each value of \( c \) there exist finite positive constants, \( B_{\text{min}} \) and \( B_{\text{max}} \) such that

\[ B_{\text{min}} < |B_n(\Sigma)| < B_{\text{max}} \]  

(2.2)

for all \( n \) and \( \Sigma \). The analysis is based upon the following identity:

\[ \Delta_{n-1}(S(\Sigma)) = g(z(\Sigma)) - g(z(\Sigma) - \Delta_n(\Sigma)). \]  

(2.3)

For large \( n \), the separation, \( \Delta_n \), is very small. Hence, we can safely replace the difference in eq. (2.3) by a derivative. Thus we can rewrite (2.3) as

\[ \Delta_{n-1}(S(\Sigma)) = \frac{dg(\Sigma)}{\Delta_n(\Sigma)} \left(1 + E_n(\Sigma)\right) \]  

(2.4)

where \( E_n(\Sigma) \) is the small error in the estimate. Hence, eq. (2.1) implies

\[ B_{n+1}(\Sigma) = B_n(S(\Sigma)) \left(1 + E_n(\Sigma)\right) \]  

(2.5)

For any fixed value of \( n \), \( \ln |B_n(\Sigma)| \) will certainly be bounded just so long as \( z = 0 \) is not an element of the Julia set. (This can happen [5], but we simply do not include this case.) Then, for sufficiently large \( n \),

\[ |E_n(\Sigma)| < 2|\Delta_n|/|z|^n_{\text{min}} = 2A2^{-n\sigma}|z|^n_{\text{min}} \]  

(2.6)

where \( |z|^n_{\text{min}} \) is the minimum value of \( |z| \) for \( z \) in the Julia set. Since the right hand side of (2.6) is a finite sum over \( n \), we have proven the uniform bound of \( B_n(\Sigma) \) given by eq. (2.2).

It is a direct consequence of eq. (2.2) that the two partition functions \( Z(\tau, \text{SET}_n) \) and \( \Gamma_n(\tau, \text{SET}_n) \) have exactly the same behavior under a limiting process of the form (1.10), since the logarithm of their ratio is bounded and the limiting process will eliminate the effect of any finite value of the ratio \( Z_n/\Gamma_n \). Hence, if the limit as \( n \rightarrow \infty \) of \( \ln \Gamma_n/(\ln \Gamma_n) \) exists and is independent of the choice of \( \text{SET}_n \), \( q(\tau) \) will be uniquely defined and equally calculable from \( Z_n \) or \( \Gamma_n \).

The limit certainly exists for the case in which \( \text{SET}_n \) is \( \text{FIX}_n \). The only problem is what will happen when the distribution of the order \( n \) or higher digits in \( \text{SET}_n \) differs from that in \( \text{FIX}_n \). However, a change in higher order digits will little affect \( g(\Sigma) \). In fact,

\[ \frac{|dg(C_{n+1}(\Sigma))/g(\Sigma)|}{|dg(\Sigma)/g(\Sigma)|} - 1 < 242^{-n\sigma}|z|^n_{\text{min}} \]  

(2.7)

for large enough \( m \). As a consequence, the distribution of high order digits cannot substantially affect \( (\ln \Gamma_n/(\ln \Gamma_n)) \), if \( n \) is large enough. In the end, \( (\ln \Gamma_n/(\ln \Gamma_n)) \) will, for large \( n \), be identical to \( (\ln \Gamma_n/(\ln \Gamma_n)) \), and the limit defining \( q(\tau) \) will be uniquely defined.

#### 2.2. Scaling Properties

The fractal nature of the Julia set is reflected in the simple scaling properties of \( \Gamma_n(\tau, \text{SET}_n) \), i.e., that this quantity goes to an exponential behavior in \( n \) for large \( n \). One way of seeing why this occurs is to specialize to the case in which \( \text{SET}_n \) is \( \text{FIX}_n \) and write \( \Gamma_n \) as a sum over symbol sequences, viz:

\[ \Gamma_n(\tau) = \sum_{\Sigma \in \text{FIX}_n} \frac{|dg(\Sigma)|^n}{2} \frac{|dg(S(\Sigma))|^n}{2} \ldots \frac{|dg(S^{-n}(\Sigma))|^n}{2}. \]

To display the \( n \)-dependence clearly, we write this result as a matrix multiplication, using as a basic matrix

\[ \langle \Sigma|M|\Sigma' \rangle = \delta_{\Sigma,S(\Sigma)}|dg(\Sigma)|. \]  

(2.9)

Then, eq. (2.8) becomes a statement involving the \( n \)th power of the matrix \( M \),

\[ \Gamma_n(\tau) = \sum_{\Sigma \in \text{FIX}_n} \langle \Sigma|M^n|S^n(\Sigma) \rangle/2^n. \]

Notice that because \( \text{FIX}_n \) is a set of periodic points of \( g^n \), \( S^n(\Sigma) = \Sigma \) so that our expression finally reduces to a trace:

\[ \Gamma_n(\tau) = \text{trace } M^n/2^n. \]  

(2.10)

If \( M \) were a matrix of fixed dimension, say 10 by 10, then \( \Gamma_n(\tau) \) would be a sum of exponentials and we would have our desired scaling result. But \( M \) is an \( n \) by \( n \) matrix. However, because \( g(\Sigma) \) depends only very weakly upon the high order digits in the symbol sequence, \( M \) behaves under a trace as if it were a lower order matrix. In particular, if we are willing to neglect the effect of all digits beyond the \( p \)th in the symbol sequence in \( g(\Sigma) \) we can simply truncate \( M \) by neglecting all the digits beyond the \( p \)th. Thus, \( M \) may be truncated down to \( \rho M \), a \( 2^p \) by \( 2^p \) matrix. Using this matrix, we can write \( \Gamma_n(\tau) \) as a sum of exponentials

\[ \Gamma_n(\tau) = \sum_{\tau} \langle \rho M^n \rangle/2^n \]  

(2.11)

where the \( \rho M \) are the eigenvalues of \( \rho M \). Thus, we have shown why equation (1.11) is true.

There is another form of scaling based upon a sequence of refinements of the Julia set. Focus upon \( \Delta_n(\Sigma) \), which is the vector separation between two neighboring elements of the Julia set. Define two “daughter” separations of this “mother” as \( \Delta_{n+1}(\Sigma) \) and \( \Delta_{n+1}(C_r(\Sigma)) \). We want to investigate the ratio of daughter to mother as

\[ R_n(\Sigma) = \frac{\Delta_{n+1}(\Sigma)}{\Delta_n(\Sigma)}. \]  

(2.12)

From our definition (2.1) this ratio takes the form

\[ R_n(\Sigma) = B_n(\Sigma) \langle B_n(\Sigma) g(S^n(\Sigma)) \rangle. \]  

(2.13)

We wish to analyze expression (2.13) in the limit of a large \( n \).

Equation (2.5) may be iterated to give

\[ B_n(\Sigma) = B_0(S^n(\Sigma))/\prod_{k=0}^{n-1} \left(1 + E_k(S^{n-1-k}(\Sigma))\right). \]  

(2.14)

Notice now that, for large \( n \), our ratio \( R_n(\Sigma) \) depends very weakly upon the lower order digits in the symbol sequence \( \Sigma \). In fact, the main dependence is upon the digits close to the \( n \)th.

To represent this behavior, we define a two sided symbol
sequence
\[ \Xi = \ldots \bar{e}_n e_0, e_1, e_2 \ldots \]
(2.15.a)
and the notation
\[ \Xi_n(e_0, e_1, e_2, \ldots) = \ldots \bar{e}_{n-2} \bar{e}_{n-1} e_n \bar{e}_{n+1} \ldots \]
(2.15.b)
For sufficiently large \( n \), then, we can write
\[ B_n(\Sigma) = B(\Xi_n(\Sigma)) \]
(2.16)
to represent its dependence upon symbols near \( e_n \). The daughter to mother ratio also depends upon this part of the sequence and can be written as
\[ R_n(\Sigma) = B(\Xi_{n-1}(\Sigma))/[B(\Xi_n(\Sigma)) \, \text{d}g(S^*(\Sigma))] \].
(2.17)
Equation (2.17) is an alternative expression of the scaling properties of our system. Here \( \Sigma \) describes the position of a point in the Julia set in the complex plane. Equation (2.17) states that the high-order daughter to mother ratio at all points in the complex plane is precisely the same and depends upon the behavior of the symbol sequence’s high order digits.
Notice that we can use this ratio to rewrite the separation as
\[ \Delta_n(\Sigma) = R_n(\Sigma) R_{n-1}(S(\Sigma)) \ldots R_1(S^*(\Sigma)). \]
(2.18)
Since \( \Delta_n(S^*(\Sigma)) \) is of order unity, we can get an asymptotic evaluation of the partition function as
\[ Z_n(\tau, \text{SET}_n) \sim \langle |R_n(\Sigma) R_{n-1}(S(\Sigma)) \ldots R_1(S^*(\Sigma))|^\tau \rangle_{\text{SET}_n}. \]
(2.19)
If we apply the same matrix device as before, replacing \( \text{SET}_n \) by \( \text{FI}X_n \), we can evaluate this expression as
\[ Z_n(\tau, \text{FI}X_n) \sim \text{trace } K^\tau/2^n \]
(2.20a)
where the new matrix \( K \) is defined by
\[ \langle \Sigma | K | \Sigma' \rangle = \langle B(\Xi_n(\Sigma)) | \Sigma | M | \Sigma' \rangle / |B(\Xi_n(\Sigma))|^\tau. \]
(2.20b)
Of course, the trace in (2.20) is exactly the same as the one in Equation (2.10) since the two matrices are similarity transforms of one another.

3. Examples
In order to make the calculations outlined visualizable, I describe the calculation of \( z(\Sigma) \) for small values of \( c \) and for large. Start with the simpler case:

3.1. The limit of large \( c \)
\[ p = \sqrt{-c} \]
(3.1)
via any convention one desires for the square root. Choose to write the inverse mapping for \( z' = g(z) \) as
\[ z = \pm p \sqrt{1 + z'/p^2} \]
(3.2)
where one defines the root via any smooth branch line which does not pass through elements of the Julia set.

Now, define \( z(\Sigma) \) recursively with the aid of equation (3.2)
\[ z(e_0, e_1, e_2, \ldots) = e_0 \, p \sqrt{1 + z(e_1, e_2, \ldots)/p^2}. \]
(3.3)
From Eq. (2.3), one discovers that for small \( p^2 \), \( z(\Sigma) \) can be expanded in a power series in \( 1/p \) as
\[ z(\Sigma) = e_0 \, p + \frac{e_0 e_1}{2} + \frac{e_0 e_1 e_2}{4p} - \frac{e_0}{8p} + O(1/p^3) \]
(3.4)
Notice how the successive terms involving higher order symbols have higher powers in \( 1/p \) so that \( z(\Sigma) \) depends mostly upon the lower-order symbols.

3.2. Small values of \( c \)
In this case, we use elements of the Julia set as \( z = U(t) \) where \( t \) is continuously varying and \( U(t) \) is a periodic function of \( t \):
\[ U(t + 1) = U(t) \]
(3.5)
The action of \( g \) upon \( U(t) \) is to change \( t \) to \( 2t \):
\[ g(U(t)) = U(2t). \]
(3.6)
At \( c = 0 \), \( U(t) \) is trivial:
\[ U(t) = W^{-1}(t) = e^{2\pi i t}. \]
(3.7)
For \( c \neq 0 \), one can write an equation for \( U(t) \) via the definition
\[ U(t) = W^{-1}(t)[1 + cU_1(t)] \]
(3.8)
and find
\[ 2U_1(t) = -W^2 + U_1(2t) - c[U_1(t)]^2. \]
(3.9)
To lowest order
\[ U_1(t) = -\sum_{k=0}^{\infty} \frac{W^{2k+1}}{2^k} - \frac{c}{2} \sum_{k, \ell = 0}^{\infty} \frac{W^{2k+2\ell}}{2^{k+\ell+1}}. \]
(3.10)
Notice that we can describe \( U(t) \) via a symbol sequence
\[ z(\Sigma) = U(t(\Sigma)) \]
where
\[ t(\Sigma) = \sum_{i=0}^\infty e_i \frac{1}{2^{i+1}}. \]
(3.11)
The only difference between the case \( c \to 0 \) and \( c \to \infty \) is that for \( c \to 0 \) we must disallow the symbol sequences which end in an infinite string of 1’s since
\[ t(1, 1, 1, 1, 1, 1, 1, \ldots) = t(-1, -1, -1, -1, \ldots) + 1. \]
(3.12)
Once again \( z(\Sigma) \) depends mostly upon the lower order elements of the symbol sequence.

3.2. Partition function calculations
Rewrite expression (1.5) using the fact that \( dg'/dz \) is a product of \( n \) factors \( dg = 2z \). Thus we have
\[ \Gamma_2(t) = \left( \prod_{j=0}^{n-1} \text{d}g(S_j(\Sigma)) \right)^t \bigg|_{\Sigma \in \text{FI}X_n}. \]
(3.13)
To the very lowest order in a large-\( c \) expansion, we can replace \( |z| \) by \( |p| \) as in eq. (3.4) and find the not-too-interesting result
\[ \Gamma_2(t) = \langle |2p|^t \rangle = |2p|^t. \]
(3.14)
This result implies that \( (q - 1) = t \ln |1/p| \) or that
\[ D_q = t/(q - 1) = \ln (2)/\ln (2|p|). \]
Here as \( |p| \to \infty \) the dimension of the set goes to zero. To next order, hold on to one more term in the expansion (2.4) and find that
$\Gamma_n(e) = \left< 2p^{e[\Sigma \in \text{FIX}_n} \right> \prod_{j=0}^{n-1} \left| 1 + \frac{e_j}{2p} \right|^2$. 

Since $|p|$ is large, one can replace the power of $(1 + e/2p)$ by an exponential and thereby obtain

$\Gamma_n(e) = \left| 2p \right|^n \exp \left< \sum_{j=1}^{n} \left( \Re \left( \frac{e_j}{2p} \right) \right) \right>$. 

Because each $e_j$ independently takes on the values $\pm 1$, the average is easily computed to be

$\Gamma_n(e) = \left| 2p \right|^n \left\{ \cosh \left( \Re \left( 1/2p \right) \right) \right\}^n$. (3.15)

Notice that expression (3.15) is of the form (1.1), i.e., an exponential with prefactor unity. The end result is

$q(t) = 1 + \left\{ \ln \left( \Lambda_+ + \Lambda_- \right) / 2 \right\} \ln 2$

where

$\Lambda_\pm = \left| 2p \right| e^{\pm \Re \left( 1/2p \right)}$

is the Floquet multiplier at the two unstable fixed points. Notice the relation between (3.15) and the two-scale factor Cantor set [6, 7].

Analogous calculations can be done as $e \to 0$. In that case (FIX)$_0$ has all period $\Sigma$'s of the form:

$e_0, e_1, e_2, \ldots, e_{n-1}, e_0, e_1, e_2, \ldots, e_{n-1}, \ldots$

except the $\Sigma$ in which all $e$'s equal 1. Then $\Gamma_n(t)$ can be calculated to be

$\Gamma_n(t) = \left< \prod_{j=0}^{n-1} 2U(\Sigma \left( S' \Sigma) \right)) \right>^{t}$. (3.17)

When (3.10) is used to expand this expression to the lowest non-trivial order in $e$ one finds

$\Gamma_n(t) = 2^n \left< \prod_{j=0}^{n-1} \left| 1 + e \sum_{\ell=1}^{\infty} W^{2e_{\ell+1}}/2^k \right|^2 \right> \Sigma \in \text{FIX}_n$. (3.18a)

The expression $2^n t^k$ arises because each shift operation doubles $t$ and squares $W$. Once again, we exponentiate to find

$\Gamma_n(t) = 2^n \exp \left[ -\Re \left( c \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{W^{2e_{k+1}}}{2^k} \right) \right]$. (3.18b)

By using the fact that when $p$ is $2^n$, $W^p = W$, one can rearrange the average to find

$\Gamma_n(t) = 2^n \left< \exp \left[ -\frac{\tau}{2} (eY + e^* Y^*) \right] \right>$ (3.18c)

where

$Y = \sum_{j=0}^{n-1} W^{2j}$

$Y^* = \sum_{j=0}^{n-1} W^{-2j}$

and

$\left< W^k \right> = \begin{cases} 1 & \text{if } k = 0 \mod 2^n \\ 0 & \text{otherwise.} \end{cases}$

For arbitrary $ct$, this expression is hard to evaluate. However for small $ct$ one can use a cumulant expansion to find

$\Gamma_n(t) = 2^n \exp \left[ -\frac{\tau^2}{4} cc^* \left< XY^* \right> \right]$

so that

$q(t) = 1 + \tau - cc^* \tau^2 / (4 \ln 2)$. (3.19)

References


3. For an elementary discussion of symbol sequences and Julia sets see Devaney, Robert L., An Introduction to Chaotic Dynamical Systems, Benjamin/Cummings, Menlo Park, California, (1986).


5. For special c-values, the point at $z = 0$ apparently can be part of the Julia set. See the discussion of the breakdown of the Siegel domain in Widom, M., Comm. Math. Phys. 92, 121 (1983). At these c-values the majority of the conclusions of the paper are false.
