DISORDER VARIABLES AND PARA-FERMIONS IN TWO-DIMENSIONAL STATISTICAL MECHANICS

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It is shown that "clock" type models in two-dimensional statistical mechanics possess order and disorder variables $\phi_n$ and $\chi_m$ with $n$ and $m$ falling in the range $1, 2, \ldots, p$. These variables respectively describe abelian analogs to charged fields and the fields of 't Hooft monopoles with charges $q = n/p$ and topological quantum number $m$. They are related to one another by a dual symmetry. Products of these operators generate, via a short-distance expansion, para-fermion operators in which rotational symmetry and the internal symmetry group are tied together. The clock models in two dimensions are shown to be an ideal laboratory where these ideas have a very simple realization.

1. Introduction

It has become quite commonplace for concepts to move up and back between statistical physics and field theory. This paper is concerned with elaborating an example from statistical physics which might perhaps illuminate in a simple context some ideas which have been employed in particle physics. In particular, we study some fields which appear (at least) superficially similar to those describing (fractionally) charged particles and topological excitations like 't Hooft monopoles [1,2]. The underlying symmetry groups are abelian, and the context is a two-dimensional lattice statistical mechanics model. In this relatively simple context, we can make much progress in analyzing the correlations among the basic fields.

The example will be described in two contexts. The first, the gaussian model, can be solved quite completely to give all correlations among the interesting fields. The second, the clock model, is a generalization of the Ising model to a case in which the basic variable take on, instead of the two values of the Ising variables, $p$ different values. We shall show that the same basic operators appear in the two contexts and play closely analogous roles. We also describe a continuous transformation which takes one from the solvable gaussian model to the "realistic"
A disorder operator $\chi_m$ and an order operator $\phi_\alpha$. The disorder operator resides at the plaquette with the cross. The angle $\theta$ measures their relative position. The seam of vertical segments represents the set of bonds whose interactions is shifted by $2\pi m/p$. (b, c) Definition of the range of the angle $\theta$.

Fig. 1 (a) A disorder operator $\chi_m$ and an order operator $\phi_\alpha$. The disorder operator resides at the plaquette with the cross. The angle $\theta$ measures their relative position. The seam of vertical segments represents the set of bonds whose interactions is shifted by $2\pi m/p$. (b, c) Definition of the range of the angle $\theta$.

The basic properties of the models are described here and then developed in detail in the succeeding sections. They are as follows.

(i) The models contain fields $\phi_\alpha(r)$ ($n = 1, 2, \ldots, p$) and a global abelian symmetry under which the fields transform as

$$\phi_\alpha(r) \rightarrow \phi_\alpha(r) e^{i\alpha_m n},$$

where $\alpha_m$ is $(2\pi/p)$ times an integer, $m = 1, 2, \ldots, p$, describing the transformation.

(ii) From this symmetry there arise a set of dual fields $\chi_m(r)$, with $m$ playing the role of a vortex quantum number of topological charge. There is a dual transformation—which leaves the hamiltonian invariant in form—which interchanges $\phi_\alpha$ and $\chi_m$. The field $\chi_m(r)$ may be identified as a field for topological excitations by the following operation. Take a correlation function containing a field $\phi_\alpha(r_1)$ and also $\chi_m(r_2)$. Continuously vary $r_1$ (see fig. 1) so that it moves in a complete circle about $r_2$. Then the correlation function returns to its former value, except that it is multiplied by a phase factor $\exp(i\alpha_m n)$. Hence this circuit effectively produces the symmetry operation of eq. (1.1).

(iii) Since the $\phi_\alpha$ and $\chi_m$ contain the group symmetry in a very explicit fashion, their product generates, via the operator product expansion [4, 5] new operators $\psi_{\alpha m}(r)$ which are generalized spinors or para-fermion variables. These operators are joint representations of the rotation symmetry of the two-dimensional space and also of the internal symmetry group. In particular, under $360^\circ$ rotation these

* A similar situation was found in ref. [3] in the context of the two-dimensional Ising model.
operators transform as

\[ \psi_{nm}(r) \rightarrow e^{ia_n} \psi_{nm}(r). \]  

(1.2)

Hence they have an apparent angular momentum quantum number, which is given by

\[ l_z = -2\pi \frac{mn}{p}. \]  

(1.3)

Thus, for example, we shall see that a correlation function formed from two of these operators has an angular dependence which is given by

\[ \langle \psi_{nm}(r) \psi_{-n,-m}(0) \rangle \sim \left( \frac{x + iy}{x - iy} \right)^{ia_{n\pi}}. \]  

(1.4)

To develop these properties we use the following strategy. In sect. 2 we introduce fields \( \phi_n(r) \) and \( \chi_m(r) \) in the context of a trivially solvable gaussian model. In this context we develop explicit formulas for correlations among arbitrarily large numbers of \( \phi_n \)’s and \( \chi_m \)’s. Then we generalize the gaussian model by adding large numbers of excitations in which \( n/p \) and \( m/p \) are integral. The number of these excitations are controlled by two fugacities \( y_0 \) and \( y_p \). In sect. 3, we show that the generalized model reduces at \( y_0 = y_p = 1 \) to the “clock model” or planar Potts model, which is one of the standard \( d = 2 \) statistical models. The fields \( \phi_n \) and \( \chi_m \) are then shown to have natural generalizations in this context and also very natural physical interpretations. Finally, in sect. 4 we return to the gaussian model to gain information about operator product expansions and also explicit forms for the correlations among an arbitrary number of \( \chi, \phi \) and \( \psi \) fields.

2. The gaussian model

Consider a two-dimensional square lattice and a variable \( \xi(r) \) at each site. The range of \( \xi \) is \([-\infty, +\infty]\). The gaussian model* is a system of such variables with a hamiltonian

\[ \mathcal{H}_{\text{gaussian}} = \frac{1}{2} K \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} (\xi(\mathbf{r} + \mathbf{e}_\mu) - \xi(\mathbf{r}))^2. \]  

(2.1)

The partition function for the gaussian model is given by

\[ Z_{\text{gaussian}} = \prod_r \int_{-\infty}^{+\infty} d\xi(r) \exp[-\mathcal{H}_{\text{gaussian}}]. \]  

(2.2)

*We use the definition of the gaussian model as given in ref. [6].
Alternatively we can represent the gaussian model by a set of angle-like site
variables $\theta(r) (0 < \theta < 2\pi)$ and integer-valued link variables $l_\mu(r)$. In terms of these
variables the partition function (2.2) reads [7]

$$Z_{\text{gaussian}} = \prod_r \int_0^{2\pi} \frac{d\theta(r)}{2\pi} \prod_{(r,\mu)} \sum_{l_\mu(r) = -\infty}^{+\infty} \left( \prod_l \delta_{\epsilon_\mu \Delta l_\mu(r)} \right)$$

$$\times \exp \left\{ -\frac{1}{2} K \sum_{(r,\mu)} (\Delta \theta(r) - 2\pi l_\mu(r))^2 \right\}, \quad (2.3)$$

where we have used the compact notation

$$\Delta \theta(r) \equiv \theta(r + \hat{e}_\mu) - \theta(r),$$

$$\epsilon_\mu \Delta l_\mu(r) \equiv l_\mu(r) + l_\mu(r + \hat{e}_\mu) - l_\mu(r + \hat{e}_\mu) - l_\mu(r), \quad (2.4)$$

for the lattice gradient and the lattice curl, respectively.

It is trivial to compute both the partition function (2.2) and also the correlation function

$$\left( \left( \prod_{i=1}^{N_n} \phi_{n_i}(r_i) \right) \left( \prod_{n=1}^{N_m} \chi_{m_j}(R_j) \right) \right) = \frac{Z[\{n\};\{m\}]}{Z_{\text{gaussian}}}, \quad (2.5)$$

where

$$Z[\{n\};\{m\}] = \prod_r \int_0^{2\pi} \frac{d\theta(r)}{2\pi} \prod_{(r,\mu)} \sum_{l_\mu(r) = -\infty}^{+\infty} \left( \prod_{j=1}^{N_m} \delta_{\epsilon_\mu \Delta l_\mu(r), m_j/p} \right)$$

$$\times \left( \prod_{i=1}^{N_n} \exp \left[ in_i \theta(r_i) \right] \right) \times \exp \left\{ -\frac{1}{2} K \sum_{(r,\mu)} (\Delta \theta(r) - 2\pi l_\mu(r))^2 \right\}, \quad (2.6)$$

whenever $m_i = 0 \pmod p$. If $m_i \neq 0 \pmod p$, $Z[nm]$ vanishes identically. Here $R$ labels the sites of the dual lattice.

If we want the system to have a non-integer curl, $m/p$, at some point we must shift the $l_\mu$ variables residing at links crossed by a path like the one shown in fig. 1 by an amount equal to $m/p$. Once a path is chosen it is possible to compute the
correlation function (2.5). The result is

$$\left\langle \prod_{i=1}^{N_n} \phi_n(r_i) \prod_{j=1}^{N_m} \chi_m(R_j) \right\rangle = \left( \prod_{i<j=1}^{N_n} \frac{1}{|r_i - r_j|^{n_i/2}} \right) \times \left( \prod_{i<j=1}^{N_m} \frac{1}{|R_i - R_j|^{m_j/2}} \right) \times \left( \prod_{i=1}^{N_n} \prod_{j=1}^{N_m} \exp\left[ in_i m_j \theta(r_i - R_j) \right] \right),$$

(2.7)

when the constraints

$$\sum_{i=1}^{N_n} n_i = 0, \quad \sum_{j=1}^{N_m} m_j = 0$$

(2.8)

are satisfied. This correlation function is equal to zero otherwise. The angle \(\theta(r_i - R_j)\) specifies the relative orientation of a charge \(n_i\) at \(r_i\) and a topological charge \(m_j\) at \(R_j\) and lies in the range \((-\pi, \pi)\) with the conventions shown in fig. 1. Notice that the angles are measured with a cut along the "path of shifted bonds" \(\Gamma\). Thus each time \(\phi_n\) crosses the path, it picks up a phase

$$\phi_n \rightarrow e^{i(2\pi n/m)}.$$  

(2.9)

This shows that \(\chi_m\) is a ladder operator which increases the phase by \(2\pi n/m\).

Until now we have a gaussian model, which contains only symmetry breaking (or spin wave) excitations and fixed topological charges created by \(\chi_m\). Let us define a generalized gaussian model [6] whose excitations are spin waves, charges \(N\) and topological charges \(M\). It is defined by the partition function

$$Z_{\Sigma, \Sigma, \Sigma} = \sum_{\{n_i\}} \sum_{\{m_j\}} Z[ pN; pM ] y_n^{\Sigma} y_m^{\Sigma} y_{\Sigma},$$

(2.10)

where \(Z[ pN; pM ]\) is given by eq. (2.6). The correlation functions of this model are

$$\left\langle \prod_i \phi_n(r_i) \prod_j \chi_m(R_j) \right\rangle = \frac{Z_{\Sigma, \Sigma, \Sigma}[\{n_i\}, \{m_j\}]}{Z_{\Sigma, \Sigma, \Sigma}},$$

(2.11)

$$Z_{\Sigma, \Sigma, \Sigma}[\{n_i\}, \{m_j\}] = \sum_{\{n_i\}} \sum_{\{m_j\}} Z[ n + pN; m + pM ] y_n^{\Sigma} y_m^{\Sigma} y_{\Sigma}.$$  

(2.12)
Here the fugacities $y_n$ and $y_m$ control the number of charges $N$ and topological charges $M$ present in the system. It is important that the path-crossing property mentioned above is maintained in this generalized model. The quantum number $m$ may be considered to be a textural singularity present in the system such that a complete circuit of $\phi_n$ about $m$ is a symmetry operation of the system. In the language of spin systems, like the planar model, the textural singularity $m$ is the endpoint of a domain wall (i.e., a magnetic dislocation). This wall favors a jump in the orientation of the spins equal to $2\pi m/p$.

### 3. The clock model

The clock model is a discrete version of the $XY$ model in which the spin can point only along $p$ different directions. The symmetry group of the clock model is the cyclic group of $p$ elements $Z_p$. The partition function for the clock model is given by

$$Z_p^{\text{clock}} = \sum_{\{\theta(r)\}} \sum_{\{l_{\mu}(r)\}} \exp \left\{ -\frac{1}{2} K \sum_{(r,\mu)} \left( \Delta_{\mu} \theta(r) - 2\pi l_{\mu}(r) \right)^2 \right\}, \quad (3.1)$$

where $\theta(r)$ is the (discrete) angle

$$\theta(r) = \frac{2\pi}{p} \sigma(r), \quad \sigma = 0, \ldots, p - 1. \quad (3.2)$$

The partition function of the $p$-state clock model relates very simply to that of the generalized gaussian model [7] (2.8)

$$Z_p^{\text{clock}} = \lim_{y_n, y_m \to 1} Z_p^{\text{gaussian}}. \quad (3.3)$$

The correlation functions of both models are also related through a similar expression.

As we have already pointed out in sect. 1, two kinds of operators can be defined in the clock model. Both contain the symmetry in a very explicit manner. The first type are the symmetry-breaking operators $\phi_n(r) = \exp[in\theta(r)]$ which are order variables. The second kind, $\chi_m(R)$, are the disorder variables [3] and create vortex-like excitations. The disorder variable $\chi_m(R)$ located at a dual site $R$, is introduced in the clock model by shifting the integer-valued bond variables $l_{\mu}(r)$ in (3.1) by a fractional amount $(q_{\mu}/p)(r)$ along a path $\Gamma$ (see fig. 2). The integer-valued variables $q_{\mu}(r)$ satisfy the constraint

$$\varepsilon_{\mu\nu} \Delta_{\nu} q_{\mu}(r) = m(R). \quad (3.4)$$

Thus, the presence of disorder variables $\chi_m(R)$ introduces in the system fractional
domain walls that favor a discontinuity in the orientation of the spins by an angle equal to \((2\pi/p)m\). If, for instance, two such disorder variables are present at dual sites \(0\) and \(R\), the domain wall will begin and end at these points. Notice by the way that, since this is a \(p\)-state model, the charge \(n\) of the order variable and the topological charge \(m\) of the disorder variable are only defined modulo \(p\).

The clock model has the important property of self-duality*. Under a dual transformation the coupling constant \(K\) transforms like

\[
\tilde{K} = \left(\frac{p}{2\pi}\right)^2 / K, \tag{3.5}
\]

while order and disorder operators are transformed into each other: \(\phi_n \leftrightarrow \chi_m\). For \(p < 4\) (i.e., \(p = 2, 3, 4\)) the clock models have a continuous phase transition [6, 8] at the self-dual coupling variable \(K = \tilde{K} = p/2\pi\). On the other hand, for \(p > 4\) these models are supposed to have two phase transitions of a Kosterlitz-Thouless character at critical coupling values \(K_{\text{max}}(p)\) and \(K_{\text{min}}(p)\) related by eq. (3.5)

\[
2\pi K_{\text{max}}(p) = \frac{p^2}{2\pi K_{\text{min}}(p)}. \tag{3.5'}
\]

Let us show now that the fields \(\phi_n\) and \(\chi_m\) obey commutation relations like those pointed out in sect. 1. First of all let us see that the correlation function \(\langle ||\phi_n(r)|| \chi_m(R) \rangle\) has a phase ambiguity as it did in the gaussian model. Consider a disorder variable \(\chi_m(R)\) residing at the dual site \(R\) and its string of shifted bonds along a path \(\Gamma\) going to the left of \(R\). Consider now an order variable \(\phi(r)\) at a site \(r\) neighbor to the path \(\Gamma\) but just under it. Imagine now performing a \(360^\circ\) rotation of the field \(\phi_n(r)\) about the position of the disorder field \(\chi_m(R)\) in a way that ends up just above the path \(\Gamma\) (fig. 3). Clearly the situation is almost

* Self-dual in the Kramers-Wannier sense. See ref. [9].
identical to the original one except for the fact that the path $\Gamma$ is “misplaced” relative to $\phi_n$. This problem can be easily solved just by performing a gauge transformation right at the location of $\phi_n$ as shown in fig. 2:

\[
\phi_n(r) \rightarrow \phi_n(r) \exp \left[ i \frac{2\pi}{p} mn \right],
\]
\[
q_\mu(r) \rightarrow q_\mu(r) + \Delta_\mu \alpha(r),
\]
\[
\theta(r) \rightarrow \theta(r) + \frac{2\pi}{p} m,
\]

where

\[
\alpha(r) = \begin{cases} m, & r' \neq r, \\ 0, & \text{otherwise}. \end{cases}
\]

Thus after a $360^\circ$ rotation the field $\phi_n$ has suffered a phase shift by $(2\pi/p)mn$ and its phase is thus ambiguous. This is so since $\phi_n$ represent a fractional charge equal to $n/p$ residing at $r$ and transforms like the $n$th irreducible representation of the $\mathbb{Z}_p$ group. Analogously if the disorder variable is rotated around the order variable we also end up with a phase shift. A way to understand this phase ambiguity is to say that the product of an order operator times a disorder operator requires an ordering prescription. This is precisely what we have done in fig. 1. Each time an order variable crosses a string attached to a disorder variable, it picks up a phase. Thus an order variable $\phi_n(r)$ and a disorder variable $\chi_m(R)$ do not commute with each other since the expectation value of products of these operators before and after a complete rotation by $360^\circ$ differs by a phase $\exp(i(2\pi/p)mn)$.

We can summarize the result by stating that the fields $\phi_n$ and $\chi_m$ obey the commutation relations

\[
\phi_n(r)\chi_m(R) = \begin{cases} \exp \left[ i \frac{2\pi}{p} mn \right] \chi_m(R)\phi_n(r), & (a) \\ \chi_m(R)\phi_n(r), & (b) \end{cases}
\]

where alternative (a) is realized if the order variable $\phi_n(r)$ performs a complete rotation around the disorder variable $\chi_m(R)$, and alternative (b) is realized otherwise.

It is interesting to note that ’t Hooft has found identical commutation relations in non-abelian gauge theories between Wilson loop operators and ’t Hooft disorder operators [10]. Indeed this is not surprising since the ’t Hooft commutation relations reflect the fact that the abelian group $\mathbb{Z}_p$ is the center of the non-abelian group $\text{SU}(p)$. More remarkable, however, is the close analogy between our
disorder variables and the 't Hooft operators. These operators are like disorder operators in the sense that their behavior is dual of that of the Wilson loops [10]. Furthermore, the 't Hooft operators create currents of fractionally charged 't Hooft-Polyakov magnetic monopoles in the same manner than our disorder operators create fractional topological charges (vortices). In fact, the path of shifted bonds $\Gamma$ is completely analogous to the magnetic monopoles.

Let us consider now, in the framework of the clock model, the behavior of the composite operator $\phi_n(r)\chi_m(R)$, where $r$ and $R$ are nearby positions. It is clear that since both $\phi_n$ and $\chi_m$ contain the internal symmetry in a very explicit manner their product generates, via an operator product expansion, a set of new local operators $\psi_{n,m}(r)$ which are para-fermion variables. This can be seen very easily just by rotating an operator $\phi_n(r)$ around another $\phi_n(r')$ at $r'$. The commutation relations (3.7) imply that the $\psi$'s themselves satisfy

$$\psi_{n,m}(r)\psi_{n',m'}(r') = \begin{cases} 
\exp\left\{i\frac{2\pi}{p}mn'\right\}\psi_{n',m'}(r')\psi_{n,m}(r), & (a) \\
\exp\left\{i\frac{2\pi}{p}nm\right\}\psi_{n',m'}(r')\psi_{n,m}(r), & (b) \\
\psi_{n',m'}(r')\psi_{n,m}(r), & (c)
\end{cases}$$

(3.8)

where in (a) $\psi_{n,m}$ performs a closed path around $\psi_{n',m'}$, (b) $\psi_{n',m'}$ performs a closed path around $\psi_{n,m}$ and (c) otherwise. As in eq. (3.7) these anomalous commutation relations originate in the fact that after a 360° rotation it is necessary to perform a gauge transformation to recover the original path configuration. Since the order variable $\phi_n(r)$ is not gauge invariant, a phase factor $\exp[i(2\pi/p)nn']$ arises. Therefore the composite operators $\psi_{n,m}(r)$ are joint representations of the (euclidean) rotations group in two-dimensional space and the internal symmetry group $Z_p$. In particular under a 360° rotation the operators $\psi_{n,m}$ transform as

$$\psi_{n,m}(r) \rightarrow \psi_{n,m}(r) \exp\left[i\frac{2\pi}{p}nm\right].$$

(3.9)

Hence the operator $\psi_{n,m}$ creates an excitation that carries an internal quantum number $l_z$

$$l_z = -\frac{nm}{p}$$

(3.10)

which plays the role of an effective angular momentum.
4. Operator product expansion

To gain further insight into the properties of the para-fermion operators $\psi_{n,m}(r)$ we must give a local definition of them. This will be done by means of the operator product expansion [4, 5].

The basic philosophy behind the operator product expansion is that if the distance between the operators $\phi_n(r)$ and $\chi_m(R)$ is much smaller than the correlation length (although much bigger than the lattice spacing) then their product defines a local operator $\psi_{nm}(r)$:

$$\phi_n(r)\chi_m(R) = C_{n,m}\left(\frac{R-r}{2}\right)^4\psi_{n,m}(r) + \cdots.$$ (4.1)

It is very easy to evaluate the coefficients $C_{n,m}(r)$ in the gaussian model, but what we really want is to have an expansion like (4.1) for the clock model.

So what we would like to do is to understand how the clock model (described by $y_n = y_m = 1$) is related to the gaussian model (described by $y_n = y_m = 0$). Whenever these models lie in the same universality class, i.e., have the same critical behavior, the gaussian model analysis will serve to give us considerable additional information about the clock models. Thus, to make the connection between the two models we should investigate what happens when $y_n$ and $y_m$ are increased bit by bit, starting from the gaussian case in which both are zero.

The behavior is different depending upon whether $p > 4$, $p = 4$ or $p < 4$ [6]. For $p > 4$ there exists a range of gaussian model couplings, $K_G$,

$$4 < 2\pi K_G < \frac{1}{4}p^2,$$ (4.2)

for which an infinitesimal increase in $y_n$ and $y_m$ will not change the structure of gaussian-model correlations. Technically speaking the corresponding excitations are irrelevant in the gaussian model. It is believed that this behavior persists in the generalized gaussian model for which $0 < y_n, y_m < 1$. Hence, the generalized model serves to give a smooth interpolation between the clock model and the essentially equivalent behavior of the gaussian model. That is, there exists a range of clock-model couplings between a minimum coupling $K_{\text{min}}(p)$ and a maximum coupling $K_{\text{max}}(p)$ such that if the clock-model coupling lies in the range

$$K_{\text{min}}(p) < K_{\text{clock}} < K_{\text{max}}(p),$$ (4.3)

then there exists a mapping between the gaussian model and the clock model,

$$K_G = F(K_{\text{clock}}, p),$$ (4.4)

such that when $K_G$ and $K_{\text{clock}}$ are related by eq. (4.4) the correlation functions of the two models are identical. In particular, the beginnings and ends of the two ranges (4.2) and (4.3) are connected by eq. (4.4).
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At $p = 4$, according to eq. (4.2), the range of validity of the relationship between the two models shrink to the point $2 \pi K_G = 4$. For $p < 4$, i.e., $p$ equal 2 or 3, there is no region of validity of the mapping (4.4). That is to say, the gaussian model is never stable against $n = p$ and $m = p$ excitations. On the other hand, in the region

$$4 > 2 \pi K_G > \frac{1}{4} p^2,$$  \hspace{1cm} (4.5)

the gaussian model is stable against these excitations.

Finally, for $p = 4$, there are a rich variety of connections between these clock-type models, called the Ashkin-Teller model or the 8-vertex model and gaussian model (see, e.g., Kadanoff and Brown [11], Luther and Peschel [12] and Coleman [13]). The exact connections are too complex and indeed too imperfectly known to be detailed here. The net result is a simple one: the gaussian model agrees with different aspects of clock-type models for all $p$ so that a treatment of the gaussian case can give considerable insight into the behavior of more realistic models.

Thus, we return to eq. (4.1), which can be written for the gaussian model in the form

$$\phi_n(r)\chi_m(R) = \exp \left[ i \frac{nm}{p} \alpha(\delta) \right] \psi_{n,m}(r),$$ \hspace{1cm} (4.6)

where $\delta = \frac{1}{2} (R - r)$ and $\alpha(\delta)$ is the angular position of the order variable relative to the disorder variable measured according to the convention shown in fig. 1.

Furthermore it is possible to compute the correlation function between $\psi_{n,m}$ variables:

$$G_{nm}(r_2 - r_1) = \langle \psi_{n,m}(r_1)\psi_{-n,-m}(r_2) \rangle = \frac{\exp \left[ -i (2nm/p) \theta \right]}{|r_2 - r_1|^{2x_{n,m}}}$$ \hspace{1cm} (4.7)

where

$$2x_{n,m} = \frac{n^2}{2 \pi K_G} + \frac{m^2}{p^2} 2 \pi K_G,$$ \hspace{1cm} (4.8)

where $K_G$ and $K_{clock}$ are related by eq. (4.4) and $\theta$ is the angle shown in fig. 1. Thus we find that the correlation function $G_{n,m}$ has an explicit angular dependence. In particular, if we commute to para-fermion operators (i.e., $\theta = \pi$) the phase of the correlation function $G_{n,m}$ jumps by $2 \pi nm/p$. Thus,

$$\langle \psi_{n,m}(r_1)\psi_{-n,-m}(r_2) \rangle = e^{-i (2nm/p) \theta \langle \psi_{-n,-m}(r_2)\psi_{n,m}(r_1) \rangle}.$$ \hspace{1cm} (4.9)

Therefore the operators $\psi_{n,m}(r_1)$ do obey parastatistics.
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It is also easy to get the following results:

\[
\langle \psi_{n,m}(r_1) \psi_{-n,-m}(r_2) \rangle = \langle \psi_{-n,-m}(r_1) \psi_{n,m}(r_2) \rangle, \quad (4.10a)
\]

\[
\langle \psi_{n,-m}(r_1) \psi_{-n,m}(r_2) \rangle = \langle \psi_{-n,m}(r_1) \psi_{n,-m}(r_2) \rangle, \quad (4.10b)
\]

\[
\langle \psi_{n,-m}(r_1) \psi_{-n,m}(r_2) \rangle = \frac{\exp[i(2nm/p)\theta]}{|r_2 - r_1|^{2x_{n,m}}}, \quad (4.10c)
\]

where \( \theta \) and \( x_{n,m} \) are the same quantities defined above. It is convenient to introduce a canonical* set of para-fermion operators whose correlations are real and simple. Two such canonical combinations are

\[
\Psi_{n,m}^{(0)}(r) = \frac{1}{2} \left[ \psi_{n,m}(r) + \psi_{n,-m}(r) + \psi_{-n,m}(r) + \psi_{-n,-m}(r) \right],
\]

\[
\Psi_{n,m}^{(+)}(r) = \frac{1}{2} e^{-i\pi/4} \left[ \psi_{n,m}(r) + i\psi_{n,-m}(r) - \psi_{-n,m}(r) - i\psi_{-n,-m}(r) \right].
\]

There are other combinations equally canonical but we will deal only with these two. The correlations between the canonical operators \( \Psi^{(0)} \) and \( \Psi^{(+)} \) are easy to compute:

\[
\langle \Psi_{n,m}^{(0)}(r_1) \Psi_{n,m}^{(0)}(r_2) \rangle = \frac{\cos((2nm/p)\theta)}{|r_2 - r_1|^{2x_{n,m}}}, \quad (4.12a)
\]

\[
\langle \Psi_{n,m}^{(+)}(r_1) \Psi_{n,m}^{(+)}(r_2) \rangle = \frac{\sin((2nm/p)\theta)}{|r_2 - r_1|^{2x_{n,m}}}, \quad (4.12b)
\]

\[
\langle \Psi_{n,m}^{(0)}(r_1) \Psi_{n,m}^{(+)}(r_2) \rangle = 0. \quad (4.12c)
\]

Hence different canonical operators do not correlate with each other. Moreover, eqs. (4.12) have some properties worthwhile mentioning. First it is clear that different species of canonical para-fermions (i.e., \( \Psi^{(0)} \), \( \Psi^{(+)} \) and \( \Psi^{(-)} \)) do not correlate (4.12c). Moreover, for some geometries even operators of the same type are not correlated. This happens if

\[
\theta = \frac{(2s + 1)}{nm} \frac{1}{4} p \pi, \quad s = 0, 1, \ldots \rightarrow \langle \Psi_{n,m}^{(0)}(1) \Psi_{n,m}^{(0)}(2) \rangle = 0, \quad (4.13a)
\]

and

\[
\theta = \frac{sp \pi}{2nm}, \quad s = 0, 1, \ldots \rightarrow \langle \Psi_{n,m}^{(+)}(1) \Psi_{n,m}^{(+)}(2) \rangle = 0. \quad (4.13b)
\]

* This operators are canonical in the sense that their correlations are real and simple. Notice that this definition of canonical operators is rather different from that used by Kadanoff [7] and Kadanoff and Brown [11].
For the operator $\Psi^{(0)}_{n,m}$ this situation occurs whenever $\theta = (2s + 1)\pi/4nm$ (s integer), while for $\Psi^{(+)}_{n,m}$ we must have $\theta = \pi s/2nm$ (s integer). So in the case $p = 4$ the correlation function $\langle \Psi^{(0)}_{1,1}(r_1)\Psi^{(0)}_{1,1}(r_2) \rangle$ vanishes if $\theta = \pi$. This means that $r_2$ lies to the left of $r_1$ along the $x$ axis. Analogously, $\langle \Psi^{(+)}_{1,1}(r_1)\Psi^{(+)}_{1,1}(r_2) \rangle$ vanishes if $\theta = 0$, which is, in fact, the opposite ordering.

We can gain further insight into the properties of these operators by computing expectation values of several of them. Let us compute, as an example, the following three-point function

$$G^{(0)}_{(3)} = \langle \Psi^{(0)}_{n_1,m_1}(r_1)\Psi^{(0)}_{n_2,m_2}(r_2)\Psi^{(0)}_{n_3,m_3}(r_3) \rangle,$$  \hspace{1cm} (4.14)

where $n = n_1 + n_2$, $m = m_1 + m_2$ and $(r_1, r_2, r_3)$ are three points on the plane. Using the fact that the only contributions to (4.14) come from those terms that satisfy $\sum_{i=1}^{2} n_i = \sum_{i=1}^{2} m_i = 0$ we obtain the result

$$G^{(0)}_{(3)}(r_1, r_2, r_3) = \frac{1}{2} \left[ \frac{|r_2 - r_1|^{\alpha_{21}}}{|r_3 - r_1|^{\alpha_{31}}|r_3 - r_2|^{\alpha_{32}}} \cos \phi_{123} \right],$$  \hspace{1cm} (4.15)

where

$$\alpha_{21} = \frac{n_1n_2}{2\pi K_G} + \frac{m_1m_2}{p^2} 2\pi K_G, \hspace{1cm} (4.16a)$$

$$\alpha_{31} = \frac{nn_1}{2\pi K_G} + \frac{mm_1}{p^2} 2\pi K_G, \hspace{1cm} (4.16b)$$

$$\alpha_{32} = \frac{nn_2}{2\pi K_G} + \frac{mm_2}{p^2} 2\pi K_G, \hspace{1cm} (4.16c)$$

$$\phi_{123} = \left( \frac{n_1m_2 + n_2m_1}{p} \right) \theta(r_2 - r_1) - \left( \frac{n_1m_1 + nm_1}{p} \right) \theta(r_3 - r_1)$$

$$\quad - \left( \frac{n_2m_1 + nm_2}{p} \right) \theta(r_3 - r_2). \hspace{1cm} (4.16d)$$

In the particular case when the three points fall on a line the phase $\phi$ depends only on the relative ordering of the operators. Thus if the ordering is $(1, 2, 3)$, the phase $\phi$ is equal to zero since all the $\theta$'s are, by definition, zero in this case. But if the ordering is $(1, 3, 2)$, we have

$$\theta(r_2 - r_1) = \theta(r_3 - r_1) = 0, \quad \theta(r_3 - r_2) = \pi,$$

$$\phi = - \frac{\pi}{p} (n_2 m_1 + nm_2). \hspace{1cm} (4.17)$$
Thus

\[ G^{(0)}_{123} = \frac{1}{2} \left| \frac{r_2 - r_1^{a_1}}{r_3 - r_1^{a_2}} \right| \left| \frac{r_3 - r_2^{a_2}}{r_2 - r_1^{a_1}} \right| , \]

\[ G^{(0)}_{123} = G^{(0)}_{123} \cos \left( \frac{\pi}{p} \left[ n_2 m + nm_1 \right] \right). \]

In particular if \( p = 4, n_m = n_2 = m_1 = m_2 = 1 \), we have \( G^{(0)}_{132} = -G^{(0)}_{123} \). However, we may also consider the sequence \((2, 3, 1)\) with the result

\[ G^{(0)}_{231} = G^{(0)}_{123} \cos \left( \frac{2n_1 m_1}{p} \pi \right), \]

and, if \( p = 4 \) and \( m_1 = n_1 = m_2 = n_2 = 1 \), we find \( G^{(0)}_{231} = 0 \). We also get the results

\[ \langle \Psi^{(+)}_{n,m_1}(r_1) \Psi^{(+)}_{n,m_2}(r_2) \Psi^{(+)}_{n,m}(r_3) \rangle = 0, \]

\[ \langle \Psi^{(+)}_{n,m_1}(r_1) \Psi^{(+)}_{n,m_2}(r_2) \Psi^{(+)}_{n,m}(r_3) \rangle = -\frac{1}{2} G^{(0)}_{123} \sin \phi, \]

\[ \langle \Psi^{(+)}_{n,m_1}(r_1) \Psi^{(+)}_{n,m_2}(r_2) \Psi^{(+)}_{n,m}(r_3) \rangle = +\frac{1}{2} G^{(0)}_{123} \sin \phi, \]

where \( \phi \) is defined by eq. (4.16d).

Finally we quote results for the four-point functions

\[ \langle \Psi^{(+)}_{1,1}(r_1) \Psi^{(+)}_{1,1}(r_2) \Psi^{(+)}_{1,1}(r_3) \Psi^{(+)}_{1,1}(r_4) \rangle = G^{(0)}_{1,1}(r_1, r_2, r_3, r_4), \]

\[ G^{(0)}_{1,1}(r_1, r_2, r_3, r_4) = \frac{1}{8} \left[ \frac{|r_2 - r_1| |r_3 - r_2|}{|r_3 - r_1| |r_2 - r_3|} \right]^2 \cos \phi(1, 2, 3, 4) \]

\[ + 5 \text{ permutations}, \]

where

\[ \phi(1, 2, 3, 4) = \frac{2}{p} \left\{ \theta(r_2 - r_1) + \theta(r_4 - r_3) - \theta(r_3 - r_1) \right. \]

\[ - \theta(r_2 - r_3) - \theta(r_4 - r_1) - \theta(r_4 - r_2) \right\}, \]

\[ 2x_{1,1} = \frac{1}{2\pi K_G} + \frac{2\pi K_G}{p^2}. \]

Analogously, it is possible to obtain correlations between the other canonical combinations and higher point functions as well.
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References