DISORDER VARIABLES FOR A NON-ABELIAN SYMMETRY GROUP*

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This paper is concerned with the construction of local disorder operators for two-dimensional statistical mechanical systems, and with the representation of these operators in transfer matrix language. Each internal global symmetry operation leads to a separate disorder variable. This idea is illustrated in the example of the Ashkin–Teller model, in which the symmetry operations form a non-abelian group. There are seven non-trivial disorder operators. Five of these are shown to be simply duals of various kinds of spin operators. Series analysis is used to describe the critical behavior of one of the remaining two operators. No essentially new scaling behavior is observed.

1. Introduction: symmetry and disorder variables

This paper describes how disorder variables [1–3] may be constructed and analyzed in cases in which the underlying symmetry group is non-abelian. Consider a situation in which there is a two-dimensional statistical system which is described by a set of statistical variables \( \varphi_m(r) \) at each lattice site \( r \). Let there be a group of symmetry operations \( T_\alpha \), which produce a transformed set of statistical variables according to

\[
T_\alpha: \varphi_m(r) \rightarrow \sum_n (T_\alpha)_{mn} \varphi_n(r).
\]

Then \( T_\alpha \) is termed a global symmetry of the problem if the hamiltonian of the system is left invariant when the same transformation (1.1) is performed at every lattice site. For each global symmetry (1.1), one can define a corresponding local disorder variable, \( D_\alpha(r) \). There is an ambiguity in the definition of averages involving \( D_\alpha \)'s and \( \varphi_n \)'s, which can be resolved by defining an ordering in the products of these operators. Hence \( D_\alpha D_\beta \) is not the same as \( D_\beta D_\alpha \). In fact, the multiplication table for \( D_\alpha(r) \) and \( D_\beta(r) \) is identical to the multiplication table for the groups elements, \( T_\alpha \). In an exactly similar fashion, one can define a commutation relation for the field variables and the disorder variables in the form

\[
D_\alpha(r) \varphi_m(r) = \sum_n (T_\alpha)_{mn} \varphi_n(r) D_\alpha(r).
\]

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In sect. 2, we discuss the construction of the disorder variables for one of the simplest models which have a non-abelian symmetry group: the Ashkin–Teller model. The symmetries form the dihedral group, and there are eight corresponding disorder operators. Six of these are duals of familiar Ashkin–Teller operators; two are new. These operators are most conveniently represented by using the transfer matrix language.

Sect. 3 is devoted to detailing the ordering and critical properties of these disorder operators. One operator is simply the unit operator; five more behave as duals of the familiar spin operators. This leaves two disorder operators which have properties that are unknown. High-temperature series are used to analyze one linear combination of these unknown operators. This combination apparently has the same critical behavior as the dual of the polarization operator.

2. Disorder variables and the Ashkin–Teller model

This model is defined on a square lattice using two Ising variables \( s(r) \) and \( \sigma(r) \), which each take on values \( \pm 1 \). These four states per site are conveniently written in terms of a variable \( \theta(r) \), which takes on the values \( (0, 1, 2, 3) \) times \( \frac{1}{2}\pi \). The correspondence between the two representations is given in table 1. The basic interaction in the model is a nearest neighbor bond, \( B(\theta, \theta') \), which defines the interaction between neighboring sites. If \( \langle r, r' \rangle \) denotes all such pairs of sites, the partition function is

\[
Z = \sum_{\{\theta\}} \exp \left( \sum_{\langle r, r' \rangle} B(\theta(r), \theta(r')) \right),
\]

the bond being given by

\[
B(\theta, \theta') = 2K_2 \cos (\theta - \theta') + K_4 \cos 2(\theta - \theta')
= K_2(\sigma \sigma' + s s') + K_4 \sigma \sigma' s s' .
\]

The model is then defined by two coupling constants \( K_2 \) and \( K_4 \). If \( K_4 = 0 \), we get two decoupled Ising models.

To define the disorder variables draw a path through the lattice (see fig. 1) which starts from a site on the dual lattice and extends to infinity. Notice that in a

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( s )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{2}\pi )</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \frac{3}{2}\pi )</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>
ferromagnetic case \((K_2 > 0, K_4 + K_2 > 0)\) it is energetically favorable to have the two \(\theta\)'s in a bond equal. Disorder is introduced to favor a rearrangement of \(\theta\)'s across the bond, according to \(\theta' \rightarrow T_\alpha(\theta')\) for the sites (indicated by crosses in fig. 1) which lie just below the path. Thus, replace the bonds shown \(B(\theta, \theta')\) by \(B(\theta, T_\alpha(\theta'))\). Now calculate a partition function. Bond interactions are calculated as before except for the bonds which cross the path. For those bonds, \(\theta\) variables are rearranged. If \(Z'\) is a partition function calculated by following the above prescription, an average of a disorder variable is defined as

\[
\langle D_\alpha(R) \rangle = Z' / Z,
\]

where \(R\) is the endpoint of the path and \(Z\) is the original partition function without rearrangements. A correlation function between two disorder variables is similarly defined using a path which connects two dual sites where the disorder variables sit (see fig. 2).

Each rearrangement defines a disorder variable. In the case of the Ising model, there is only one rearrangement, i.e., \(\sigma \rightarrow -\sigma\). This defines the disorder variable of Kadanoff and Ceva [1]. For the Ashkin–Teller model, there are \(4!\) rearrangements.

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Fig. 1. A path which defines a disorder variable.

Fig. 2. A path which defines a correlation function of disorder variables \(\langle D(R_1)D(R_2) \rangle\).
TABLE 2
Disorder variables in the Ashkin-Teller model

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \theta ) language</th>
<th>Spin language</th>
<th>Transformation operator, ( t_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_0 )</td>
<td>( \sigma \rightarrow \sigma )</td>
<td>( s \rightarrow s )</td>
<td>1</td>
</tr>
<tr>
<td>( O_1 )</td>
<td>( \sigma \rightarrow \sigma )</td>
<td>( s \rightarrow -\sigma )</td>
<td>( \tau_0 \tau_s )</td>
</tr>
<tr>
<td>( O_{-1} )</td>
<td>( \sigma \rightarrow -\sigma )</td>
<td>( s \rightarrow s )</td>
<td>( \tau_s \tau_0 )</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>( \sigma \rightarrow -\sigma )</td>
<td>( s \rightarrow -s )</td>
<td>( \tau_0 \tau_s )</td>
</tr>
<tr>
<td>( R_0 )</td>
<td>( \sigma \leftrightarrow s )</td>
<td>( s \rightarrow -s )</td>
<td>( \tau_0 \tau_1 )</td>
</tr>
<tr>
<td>( R_1 )</td>
<td>( \sigma \rightarrow -\sigma )</td>
<td>( s \rightarrow s )</td>
<td>( \tau_0 \tau_1 )</td>
</tr>
<tr>
<td>( R_{-1} )</td>
<td>( \sigma \rightarrow -\sigma )</td>
<td>( s \rightarrow -s )</td>
<td>( \tau_0 \tau_1 )</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>( \sigma \leftrightarrow s )</td>
<td>( s \rightarrow -s )</td>
<td>( \tau_0 \tau_1 )</td>
</tr>
</tbody>
</table>

However, sensible disorder variables are defined only by rearrangements which leave bond interactions unchanged. These rearrangements are global symmetries of the system, i.e.,

\[
B(T_n(\theta), T_n(\theta')) = B(\theta, \theta').
\] (2.4)

There are eight such invariance transformations: \( O_p \) which are rotations \( \theta \rightarrow \theta + \frac{1}{2} \pi p, \) \( p = 0, \pm 1, 2 \) and the rotations plus reflections, \( R_p, \theta \rightarrow -\theta + \frac{1}{2} \pi p. \) These operations are listed in table 2.

The disorder variables are defined with the aid of particular paths, which we interpret as Bloch walls. However, any correlation function involving only a single kind of disorder variable is independent of the path chosen. To see this, consider the path represented by the broken line in fig. 3. Then let us focus on the bond interactions which involve \( \theta \) variables at the site \( O \) in the calculation of the partition function \( Z'. \) Look at the quantity

\[
\tilde{Z} = \sum_{\theta_0=0, \pi/2, \pi, 3\pi/2} \exp \left[ B(\theta_{0}, T(\theta_{0})) + B(\theta_{0}, \theta_{0}) + B(\theta_{0}, \theta_{0}) + B(\theta_{0}, \theta_{0}) \right],
\] (2.5)

which is the part of \( Z' \) which involves the summation over \( \theta_{0}, \) and \( \theta_{0}, \theta_{0}, \theta_{0}, \theta_{0}, \) and \( \theta_{0}, \) \( \theta_{0}, \theta_{0}, \) and \( \theta_{0}, \theta_{0}, \theta_{0} \) and \( \theta_{0}, \) are \( \theta \) variables defined on sites O, A, B, C, and D, respectively. The function \( T \) represents a rearrangement. In the first term in the exponential, \( \theta_{0} \) is rearranged to

Fig. 3. Two alternative paths which give the same contributions to averages of disorder variables.
The problem is that the bond OA crosses the path. The rearrangement $T$ leaves the bond interaction invariant. The invariance (2.4) enables one to write expression (2.5) in the form

\[
\tilde{Z} = \sum_{\theta_0=0,\pi/2,\pi,3\pi/2} \exp \left[ B(\theta_0, T(\theta_0)) + B(T(\theta_0), T(\theta_0)) \right. \\
+ \left. B(T(\theta_0), T(\theta_0)) + B(T(\theta_0), T(\theta_0)) \right] \\
= \sum_{\theta'_0=0,\pi/2,\pi,3\pi/2} \exp \left[ B(\theta_0, \theta'_0) + B(\theta'_0, T(\theta_0)) \right. \\
+ \left. B(\theta'_0, T(\theta_0)) + B(\theta'_0, T(\theta_0)) \right],
\]

(2.6)

where $\theta'_0 = T(\theta_0)$. The second expression of eq. (2.6) shows that the bond OA is now treated normally, but the bonds OB, OC and OD are modified. This means that the original path is deformed to the dotted line in fig. 3 without changing the value of $\tilde{Z}$. By applying the above procedure successively, it is shown that all paths give the same result. Therefore, the disorder variables can be considered to be local variables. The proof given above is general and can be applied to other two-dimensional lattice problems. Each symmetry of a system defines one kind of disorder variable.

In the case of the Ashkin-Teller model, the symmetry is the dihedral group, $D_4$. The multiplication table for the group is simply derived from the statement that the $O_\alpha$'s are rotations ($\rho$ being defined modulus 4) and that $R_\alpha$ is a reflection. Then $R_\rho$ is a composition of these operations defined as $R_\rho = O_\rho R_\rho$. The multiplication rules which result are

\[
O_\rho O_\sigma = O_{\rho+\sigma}, \quad R_\rho O_\sigma = R_{\rho-\sigma}, \\
R_\rho R_\sigma = O_{\sigma-\rho}, \quad O_\rho R_\sigma = R_{\rho+\sigma}.
\]

(2.7)

The disorder variables are most conveniently written when the Ashkin-Teller model is expressed in the transfer matrix language [4]. Here in a system which contains $N$ rows and $M$ columns, see fig. 4, the partition function is written as

\[
Z = Tr V^N.
\]

Fig. 4. A path which defines a disorder variable.
Rows and columns of the transfer matrix $V$ are specified by configuration of $\theta$'s in one row of the two-dimensional square lattice whose size is $N \times M$. Since $\theta$ can take four different values, the dimension of the transfer matrix is $4^M \times 4^M$. To calculate an expectation value of a disorder variable, we need a modified partition function $Z'$. Since this is independent of a path, we can choose a path shown in fig. 4 and

$$Z' = \text{Tr} \, V^{N-"D_\alpha(x)\, V"}.$$  \hspace{1cm} (2.8)

Here, $D_\alpha(x)$ is a matrix with matrix elements

$$\langle \theta | D_\alpha(x) | \theta' \rangle = \left( \prod_{j \leq x} \delta_{\theta(j), \theta(j)} \right) \times \left( \prod_{j > x} \delta_{\theta(j), \tau_{\alpha}(\theta(j))} \right).$$  \hspace{1cm} (2.9)

It is convenient to rewrite these expressions in a spin operator representation. Convert $\theta(j)$ into an expression involving a pair of commuting matrices $\sigma(j)$ and $s(j)$, which each have eigenvalues $\pm 1$ via

$$e^{i\theta(j)} = \frac{s(j) + \sigma(j)}{2} + i\frac{\sigma(j) - s(j)}{2}.$$  \hspace{1cm} (2.10)

This is equivalent to the rewriting given in table 1. Define also operators $\tau_{\sigma}(j)$, $\tau_{s}(j)$ and $\tau_{s}(j)$, each with square unity, which act respectively to flip the sign of $\sigma$, $s$, and to interchange $\sigma$ and $s$. If $|\sigma', s'\rangle$ is an eigenstate of $\sigma$ and $s$, then $\tau_{\sigma}$, $\tau_{s}$ and $\tau_{s}$ are defined via

$$\tau_{\sigma}|\sigma', s'\rangle = |\sigma', s'\rangle, \quad \tau_{s}|\sigma', s'\rangle = |\sigma', -s'\rangle, \quad \tau_{s}|\sigma', s'\rangle = |s', \sigma'\rangle.$$  \hspace{1cm} (2.11)

Thus eq. (2.9) may be rewritten in an operator notation as

$$D_\alpha(x) = \prod_{j > x} \tau_{\alpha}(j).$$  \hspace{1cm} (2.12)

For example, from column three of table 2, we find that $R_0$ has the effect $\sigma \leftrightarrow s$, and consequently the corresponding transformation operator is $t_{\alpha}(j) = \tau_{s}(j)$. These kinds of data are filled in in column four of table 2.

For completeness, we write down the form of the transfer matrix, namely

$$V = V_1 \, V_2,$$

$$V_1 = \prod \exp \left\{ K_\alpha^x [\sigma(j) \sigma(j + 1) + s(j) s(j + 1)] + K_\alpha^y \sigma(j) \sigma(j + 1) s(j) s(j + 1) \right\},$$  \hspace{1cm} (2.13)

$$V_2 = \prod \left[ e^{2K_\alpha^x + K_\alpha^y \tau_{\alpha}(j) + \tau_{\alpha}(j) + \tau_{s}(j)} e^{-K_\alpha^y + \tau_{\alpha}(j) \tau_{s}(j)} e^{-2K_\alpha^x + K_\alpha^y} \right].$$

Here $K_\alpha^x$ and $K_\alpha^y$ describe coupling constants in the $x$ and $y$ directions.
This model has a dual symmetry [5], namely an invariance of the partition function under the interchange of operators

$$\sigma(M)\sigma(j) \leftrightarrow \prod_{k>j} \tau_{\sigma}(k) = \prod_{k>j} t_{R_1}(k) \equiv R_1(j),$$

$$s(M)s(j) \leftrightarrow \prod_{k>j} \tau_{s}(k) = \prod_{k>j} t_{R_{-1}}(k) \equiv R_{-1}(j),$$

$$e^{i\theta(\sigma(j) - \theta(M))} \leftrightarrow \prod_{k>j} t_{O_p}(k) = O_p(j).$$

(2.14)

We henceforth assume that the lattice is strongly coupled for large \(j\) and that \(\sigma(M) = s(M) = e^{i\theta(M)} = 1\). The transformations (2.14) will leave all correlation functions invariant if one also changes the coupling according to the prescription \(K_{x}, K_{y}, K_{x}', K_{y}' \rightarrow \tilde{K}_{x}, \tilde{K}_{y}, \tilde{K}_{x}', \tilde{K}_{y}'\), with

$$\sinh 2K_{x}' \sinh 2K_{y}' = \exp[-2(K_{x} + K_{y}')],$$

$$\sinh 2K_{x}' \sinh 2K_{y}' = \frac{\sinh 2K_{x}'}{\sinh 2K_{y}'}.$$

(2.15)

The same equation holds when \(x\) and \(y\) are interchanged.

Given this invariance, which takes the Ashkin–Teller model for \(T > T_c\) into the same problem for \(T < T_c\), we have considerable information about the disorder variables \(O_1, O_{-1}, O_2, R_1\) and \(R_{-1}\). Eqs. (2.14) say that under the duality transform

$$\sigma(j) \leftrightarrow R_1(j),$$

$$s(j) \leftrightarrow R_{-1}(j),$$

$$e^{i\theta(\sigma(j) - \theta(M))} \leftrightarrow O_p(j).$$

(2.16)

Hence all our disorder variables are simply duals of the familiar order variables. In fact since \(e^{i\theta}\) is given by \(\frac{1}{2}[\sigma + s \pm i(\sigma - s)]\), \(R_1\) and \(R_{-1}\) are exactly related to \(O_1\) and \(O_{-1}\) via

$$R_1 + R_{-1} = O_1 + O_{-1},$$

$$R_1 - R_{-1} = -i(O_1 - O_{-1}).$$

(2.17)

Thus we find, for example, that the critical indices of \(R_1, R_{-1}\) and \(O_1\) and \(O_{-1}\) are all exactly the same as those of the order parameter, while the critical index of \(O_2\) is equal to that of the polarization, \(\alpha\). These statements are true along the critical line on which the Ashkin–Teller model is self-dual.

The remaining variables \(R_0\) and \(R_2\) remain to be understood. The next section is devoted to these variables.
3. The rotation disorder variables

To gain a further insight into the behavior of the disorder variables, we use the definition

$$D_\alpha(j) = \prod_{k \gg j} t_\alpha(k)$$  \hspace{1cm} (3.1)

to write a commutation relation for the disorder variables, namely

$$D_\alpha(j)D_\beta(k) = D_\gamma(\beta;\alpha)(k)D_\alpha(j), \quad \text{for} \ k \gg j.$$  \hspace{1cm} (3.2)

Here $\gamma(\beta \mid \alpha)$ is defined via a conjugacy relation in the group multiplication table, i.e.,

$$T_{\gamma(\beta \mid \alpha)} = T_\alpha T_\beta T_\alpha^{-1}.$$  \hspace{1cm} (3.3)

If $k$ and $j$ are far separated, eq. (3.2) gives a statement about averages of disorder variables,

$$\langle D_\alpha \rangle \big( \langle D_\beta \rangle - \langle D_{\gamma(\beta \mid \alpha)} \rangle \big) = 0.$$  \hspace{1cm} (3.4)

In the high temperature domain $K_2 \approx 0$, $K_4 \approx 0$ all the averages are non-zero. Hence eq. (3.4) implies that the average of $D_\beta$ and $D_\alpha$ will be equal wherever $T_\beta$ and $T_\alpha$ fall in the same class. Thus,

$$\langle O_1 \rangle = \langle O_{-1} \rangle, \quad \langle R_1 \rangle = \langle R_{-1} \rangle, \quad \langle R_0 \rangle = \langle R_2 \rangle.$$  \hspace{1cm} (3.5)

For low temperatures $K_2 \rightarrow \infty$, $(K_2 + K_4) > 0$ the duality implies that the average of $R_1$, $R_{-1}$, $O_1$, $O_2$ and $O_{-1}$ are all zero. However, no information is given about $R_0$ and $R_2$ by this argument.

A relation like (3.2) can be written for the commutation of order and disorder variables [2]

$$e^{i\theta j(k)} D_\alpha(j) = D_\alpha(j) e^{i\theta j(k)}.$$  \hspace{1cm} (3.6)

At $k$ and $j$ set far apart, eq. (3.6) implies

$$\langle D_\alpha \rangle \big( \langle e^{i\theta j} \rangle - \langle e^{i\theta j(k)} \rangle \big) = 0.$$  \hspace{1cm} (3.7)

Go to low temperatures and consider successively cases in which $\theta = \frac{1}{2}\pi \ (0, 1, 2, 3)$. We then find that $\langle R_0 \rangle \theta$ is zero if the ordered state has $\theta = \pm\frac{1}{2}\pi$ and $\langle R_2 \rangle \theta$ is zero when $\theta = 0$ or $\pi$. Since rotations of $\theta$ will convert $R_0$ into $R_2$, we conclude that for $T < T_c$ all the non-zero averages are equal:

$$\langle R_0 \rangle \theta = \langle R_2 \rangle \theta + \pi/2, \quad \text{for} \ T < T_c.$$  \hspace{1cm} (3.8)

These low-temperature data are then consistent with the statements that in the low-temperature domain

$$R_0(j) = \frac{1}{2}(1 + \sigma(j)s(j)),$$

$$R_2(j) = \frac{1}{2}(1 - \sigma(j)s(j)).$$  \hspace{1cm} (3.9)
To gain additional data we turn to high-temperature series derived in the anisotropic limit in which \( K^2 \) and \( K^4 \) go to \( \infty \) while \( K^2 \) and \( K^4 \) go to zero. In this limit, \( \ln V \) can be expressed in a very simple form. It is proportional to a Hamiltonian operator, \( H \):

\[
H = \sum \limits_j \left[ \tau_\sigma(j) + \tau_\sigma(j) + \lambda \tau_\sigma(j) \tau_\sigma(j) \right] + \beta[\sigma(j)\sigma(j+1) + s(j)s(j+1)] \\
+ \lambda s(j)s(j+1)\sigma(j)\sigma(j+1)] .
\] (3.10)

We have used this Hamiltonian \([4]\) to calculate the properties of the Ashkin–Teller model by deriving high-temperature \((\beta \to 0)\) series. The linked cluster method by Kadanoff and Kohmoto \([6]\) was used in the calculation. In the region \(-\frac{1}{2} < \lambda < 1\), there is apparently one critical value of \( \beta \), namely \( \beta = 1 \). On the line, \( \beta = 1 \), \( H \) is self-dual. We report here analysis of eleventh order series for “magnetizations”, \( \langle R_0 \rangle = \langle R_2 \rangle \), and also the “susceptibility” for the combination

\[
R = \frac{1}{2}(R_0 + R_2) .
\] (3.11)

The susceptibility is defined by

\[
\chi_R = \frac{\partial \langle R \rangle_{H+k\sum \sigma R(j)}}{\partial h} \bigg|_{h=0} .
\] (3.12)

The results are compared with those for the rotation disorder operator \( \langle O_2 \rangle \). The magnetizations for the two operators are expressed as

\[
\langle R \rangle = 1 + \sum \limits_{k=2} a_R(k)\beta^k ,
\] (3.13)

\[
\langle O_2 \rangle = 1 + \sum \limits_{k=2} a_0(k)\beta^k .
\] (3.14)

From the series analysis, we learn that the values of \( a_R(k) \) are very close to \( \frac{1}{2}a_0(k) \) (see table 3). The different series for the two susceptibilities are also close to

\[
\text{Table 3}
\]

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>(-0.9)</th>
<th>(-0.5)</th>
<th>0</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>k ( \leq 7 )</td>
<td>( 1.30 \times 10^{-2} )</td>
<td>( 1.36 \times 10^{-2} )</td>
<td>( 3.24 \times 10^{-3} )</td>
<td>( 1.38 \times 10^{-3} )</td>
<td>( 6.77 \times 10^{-4} )</td>
</tr>
<tr>
<td>8</td>
<td>( 8.64 \times 10^{-3} )</td>
<td>( 8.69 \times 10^{-3} )</td>
<td>0</td>
<td>( 2.07 \times 10^{-3} )</td>
<td>( 1.09 \times 10^{-4} )</td>
</tr>
<tr>
<td>9</td>
<td>( -1.74 \times 10^{-3} )</td>
<td>( 1.71 \times 10^{-2} )</td>
<td>( 8.05 \times 10^{-3} )</td>
<td>( 3.17 \times 10^{-2} )</td>
<td>( 2.28 \times 10^{-5} )</td>
</tr>
<tr>
<td>10</td>
<td>( 3.66 \times 10^{-3} )</td>
<td>( 2.34 \times 10^{-2} )</td>
<td>0</td>
<td>( 5.04 \times 10^{-5} )</td>
<td>( 1.03 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

The numbers shown are \( \{a_R(k) - \frac{1}{2}a_0(k)\} / a_R(k) \), where \( k \) represents order of the series in terms of \( \beta \) and \( a_R(k) \) and \( a_0(k) \) are defined in eqs. (3.13) and (3.14). This quantity is 0 for \( k \leq 7 \).
identical, except for a factor of 4. From these results we have to conclude that the operator $R$ is not quite independent of the others for $\lambda > -\frac{1}{2}\sqrt{2}$, but

$$R = \frac{1}{2}O_2 + \frac{1}{2}.$$ (3.15)

The behavior of the series strongly suggests that, in the critical region, $R$ is not an independent operator but behaves very similarly to $\frac{1}{2}(1 + O_2)$. In fact, we hypothesize that in their critical behaviors $R_0$ and $R_2$ respectively behave as

$$R_0(j) = \frac{1 + O_2(j) + \sigma(j)s(j)}{2},$$

$$R_2(j) = \frac{1 + O_2(j) - \sigma(j)s(j)}{2}.$$ (3.16)

If this is so, none of the disorder operators show new critical indices.

We acknowledge Stephen Shenker for his assistance in the early stage of this work and for many stimulating discussions. We had a helpful discussion with Stuart Samuel. Marcel den Nijs helped us find and correct an error which arose in the course of this work.

References