DYNAMICS OF A COMPLEX INTERFACE

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Received 4 April 1990
Accepted 23 August 1990
Communicated by Y. Pomeau

We study the motion of the interface between two fluids in a pressure field. A more viscous fluid surrounds a finite region filled with a less viscous fluid whose pressure is constant. The more viscous fluid is incompressible and moves with a velocity proportional to the gradient of its pressure. The pressure jump across the interface between the fluids is proportional to the curvature of the interface. The proportionality constant is the surface tension. In the two-dimensional case the interface can be described by a complex function which is analytic in the exterior of the unit circle. The dynamics are governed by a nonlocal nonlinear equation for this function. If all the singularities of the function are situated near the origin then the nonlocal interactions contribute little to the evolution of the singularities. In this case the nonlocal evolution equation can be approximated by a local nonlinear one. We prove that this equation has solutions with certain uniform behavior as the surface tension is allowed to become vanishingly small.

1. Introduction

The Hele-Shaw problem [1–4] is one in which two fluids are confined between closely spaced glass plates. One fluid is viscous and incompressible, the other is inviscid and has constant pressure. The viscous fluid moves with a velocity which is proportional to the two-dimensional gradient of its pressure. Because of the incompressibility, the pressure of the viscous fluid is a harmonic function. The boundary condition at the interface between the two fluids is that the pressure jump is proportional to the curvature of the interface. The proportionality constant \( \tau \) is the surface tension. (This condition is known to be an oversimplification [5–8].) The interface moves with the fluid. A sink of strength \( 2\pi \) at infinity provides the boundary condition at infinity.

It is clear that this problem can be formulated as an evolution equation for the interface: The solution of Laplace's equation with boundary data \( \kappa \) is given by a boundary integral. If \( \kappa \) is the prescribed quantity, namely a linear function of the curvature of the boundary, then this integral represents the pressure. Computing the gradient of this function at the boundary one obtains another boundary integral; this last integral is proportional to the time derivative of the boundary. A local existence result [9] is obtained using this Lagrangian formulation. The difficulty with this approach is in the fact that the kernel of the integral is not expressed explicitly in terms of the boundary.

Another approach [10, 11] is as follows. Consider a conformal transformation which maps the exterior of the unit circle in the \( w \) complex plane to the domain in the \( z \) plane occupied by the more viscous fluid at time \( t \):

\[
z = f(w, t). \tag{1.1}
\]

Denote \( h \) the function

\[
h(w, t) = \partial_w f(w, t). \tag{1.2}
\]
The function $h$ obeys a complicated nonlinear integro-differential equation which we will describe below (eq. (2.9)). The main purpose of this paper is to derive and analyze a simpler differential equation (eq. (1.5)) which describes the local interactions of singularities.

In the $\tau = 0$ case the function $\log w$ is a complex velocity potential whose real part is the pressure. The functions $f$ and $h$ can be interpreted in terms of the position $z = x + iy$ and velocity $(v_x, v_y)$ of a particle of fluid with velocity potential $w$:

$$v_x - iv_y = [w h(w, t)]^{-1}. \tag{1.3}$$

The presence of a sink at infinity requires that $f$ equate asymptotically to $w$ at infinity. Given the interpretation (1.3), $h$ must have neither zeroes nor poles in the physical region (exterior of the unit circle). In this case of $\tau = 0$ the basic equation is an integrable system [10, 12]. The case in which $h$ has $n$ zeroes and $m$ poles ($n > m$) in the unit circle can be analyzed in detail [13]. If $h$ starts out as a rational function it remains so, at least until the singularities (zeroes or poles) hit the unit circle. The ordinary differential equations for these singularities can be written explicitly. When a singularity does hit the unit circle the solution breaks down (see, however, refs. [14–16]). A well-known [17] model related to the zero surface tension problem, known as DLA [18], displays extremely chaotic behavior. Furthermore, low surface tension problems exhibit chaotic behavior [19–22]. The nonzero surface tension problem is a singular perturbation of the zero surface tension one [23–28].

Now we are in a position to state our main results. We will consider the conformal transformation $f$ in the presence of surface tension and in the case when the interface is almost circular, i.e., $h$ is close to being constant on the unit circle. Because of the behavior imposed on $f$ at infinity ($w^{-1} f(w)$ is bounded) it follows that nonconstant $h$ must have singularities inside the unit circle. The structure of the complicated nonlocal equation obeyed by $h$ is such that, near these singularities, a simpler local equation is effectively obeyed. In section 3 we derive the form of this equation for the case in which the singularities lie very close to the center of the circle.

The equation is written for $g$ defined by

$$h(w, t) = r(t) g(w, t), \tag{1.4}$$

where $r(t)$ is the radius of the circular solution. This solution is obtained easily. The equation for the area $A(t)$ of a perfectly circular bubble in the presence of a sink of strength $2\pi$ at infinity is

$$\frac{dA}{dt} = 2\pi.$$

We make the convention that at the initial time this area is $\pi$. Consequently, the radius of the bubble is $r(t) = (2t + 1)^{1/2}$. Clearly, the conformal transformation for a perfectly circular bubble is $f = r(t) w$ and its derivative is $r(t)$.

At infinity $g$ is required to equal 1. The equation for $g$ is

$$r^2(t) \frac{dg}{dt} = 2 - 2g - Dg + \frac{2\tau}{r(t)} D(I - D^2) g^{-1/2}, \tag{1.5}$$

where $D = w (\partial / \partial w)$.

We study the properties of eq. (1.5). When $g(w, 0)$ is analytic outside a circle and sufficiently close to 1 in an appropriate sense, we prove that for any later time $t$, $g(w, t)$ is analytic outside another circle (which grows in time exactly as $r(t)$ does) and is uniquely defined throughout this region of analyticity. Furthermore, in this region, the maximum difference between corresponding solutions of the $\tau = 0$ and $\tau > 0$ equations vanish at least linearly with $\tau$ as $\tau$ tends to 0. The proof is done by analyzing a mixed numerical scheme which combines features of a Newton and of a backward Euler scheme. An essential ingredient in the proof is the choice of the way convergence is measured: with the right choice of norm there is no loss of domain of analyticity.
2. Equations of motion

In order to describe the equations of motion we first introduce our notation. We will deal with complex valued functions of two variables: \( w \), which is complex, and \( t \), which represents time and is real. First we will assume \( |w| = 1 \), i.e.

\[ w = e^{i\alpha} \]

with \( \alpha \in [0, 2\pi] \). The operator \( D \) is defined by

\[ Df = \frac{1}{i} \partial_\alpha f. \]  

(2.1)

Note that

\[ Df = w \partial_w f. \]  

(2.2)

The usual Hilbert transform on the circle is

\[ Hf(\alpha) = \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} \cot\left(\frac{1}{2}(\alpha - \beta)\right) f(\beta) d\beta. \]  

(2.3)

The operator \( A \) is defined by

\[ A = \frac{1}{2}(I - iH). \]  

(2.4)

If

\[ f = \sum_{k \in \mathbb{Z}} f_k e^{ik\alpha} \]  

(2.5)

then

\[ Hf = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-i \text{sgn} k) f_k e^{ik\alpha} \]  

(2.6)

and therefore

\[ Af = \frac{1}{2} f_0 + \sum_{k < 0} f_k e^{ik\alpha}. \]  

(2.7)

We need one more object which corresponds to the curvature of the interface. In the present context it is a nonlinear functional \( \mathcal{K} \) defined by

\[ \mathcal{K}(f) = |f|^{-1} \left[ 1 + \text{Re}\left( \frac{Df}{f} \right) \right]. \]  

(2.8)

The equation of evolution for the interface, phrased in terms of the values on the unit circle of the function \( h \) defined in (1.2) is (see appendix)

\[ \frac{\partial h}{\partial t} = F(h) + \tau G(h), \]  

(2.9)

where \( \tau \) is a positive constant (the surface tension) and

\[ F(h) = 2(I + D)\left[hA(|h|^{-2})\right] \]  

(2.10)

and

\[ G(h) = 2(I + D)\left[hA\left(|h|^{-2}\frac{(2A - I)}{A} \mathcal{K}(h)\right)\right]. \]  

(2.11)

If \( h \) is the boundary value of a function which is analytic in the exterior of the unit circle (and is at most constant at infinity) so are \( F(h), G(h) \). In other words, if all the Fourier coefficients of \( h \) corresponding to positive \( k \) vanish, then the same is true for \( F(h), G(h) \). If, for such \( h \), the initial function equals one at infinity so does the solution for all the later times when it is defined.

3. Dynamics of singular regions

One exact solution of eq. (2.9) is the circular interface

\[ h = r(t), \]  

(3.1)

where \( r(t) \) is the radius of a perfectly circular bubble of area

\[ \pi r(t)^2 = \pi (1 + 2t). \]  

(3.2)

This formula for the area follows from the equation of the area in the presence of a sink.

In order to study the behavior of interfaces which are initially nearly circular we perform the
change of dependent variable

\[ h = r(t) \ g \]  \hspace{1cm} (3.3)

and rescale time

\[ s = \log r(t). \]  \hspace{1cm} (3.4)

In these rescaled variables the basic equation (2.9) becomes

\[ \frac{dg}{ds} = (F-I)(g) + \tau e^{-s}G(g). \]  \hspace{1cm} (3.5)

The range of \( s \) is \([0, \infty)\) and the steady solution \( g = 1 \) of (3.5) corresponds to the circular interface \( r(t) \). The equation for infinitesimal disturbances (linearization, first variation or Gateaux differential) of (3.5) at \( g = 1 \) is

\[
\frac{d\nu}{ds} = D\nu - 4(I + D)A \text{Re} (\nu) + 2\tau e^{-s}(I + D) \\
\times AD(2A - I) \text{Re} [(D - I)\nu]. \tag{3.6}
\]

The zero-order Fourier coefficient has a trivial behavior and may be assumed to vanish. If the initial disturbance can be extended analytically to the exterior of the unit circle (physical case) then this property persists in time and (3.6) becomes

\[
\frac{d\nu_j}{ds} = [ -2I - D + \tau e^{-s}D(D^2 - I)]\nu_j. \tag{3.7}
\]

On Fourier coefficients this equation reads

\[
\frac{dv_j}{ds} = [ -2 + |j| - \tau e^{-s}|j|(|j|^2 - 1)]v_j \tag{3.8}
\]

for \( j < 0 \).

It is clear from the equation above that the \( \tau = 0 \) problem is ill-posed in the sense of Hadamard (the smaller the wavelength of the disturbance the faster it grows [29]). The presence of a positive \( \tau \) has a smoothing effect for a short time; for long times the term involving \( \tau \) becomes negligible. The nonlinear initial value problem \( (\tau > 0) \) will have short time solutions which can be extended analytically to the exterior of the unit circle. Their life-span will deteriorate with decreasing \( \tau \). If one is interested in the long time behavior of solutions and in the \( \tau \to 0 \) limit one has to consider functions whose domains of analyticity decrease with time. This is motivated by the explicit solution of the \( \tau = 0 \) equation (3.8):

\[
u(\xi, s) = e^{-2s}\nu(\xi e^{-s}, 0). \tag{3.9}\]

Until now all functions were defined on the unit circle; in (3.9) we abused notation and extended the functions to the complex plane. This general solution of the \( \tau = 0 \) case shows that if the initial disturbance is analytic in the exterior of a small region around the origin, then the solution will be defined and analytic in the exterior of a growing region. In terms of the original time variable, if we denote \( S_0 \), the complement of the domain of analyticity (singular set) then

\[ S_\tau = (2t + 1)^{1/2}S_0. \tag{3.10}\]

Therefore, for most initial data, the singular set will hit the unit circle in finite time.

We suspect that solutions to the full nonlinear problem (2.9) behave in a way which is close for small \( \tau \) to what is observed in the \( \tau = 0 \) linear case: if the initial datum is close to one and has singularities only in a small neighborhood of the origin then the solution will have singularities which move toward the unit circle with a speed which is at most the speed dictated by \( r(t) \) and does not deteriorate as \( \tau \to 0 \). In particular, the life-span of solutions with this kind of initial datum should be bounded below uniformly for small \( \tau \).

If one is interested mainly in the singularities of solutions then a considerable simplification of (2.9) is possible for functions with singular sets near the origin. Suppose a function \( h \) initially defined on the unit circle can be extended to a function which is analytic in the complement of a small neighborhood of the origin and finite at infinity. The complex conjugate \( h^* \) of this func-
tion can be extended to an analytic function in the interior of the unit circle (eqs. (A.1)–(A.2)). Therefore, near the origin where the extension of \( h \) is almost singular, the extension of \( h^* \) is continuous and hence close to the constant value it takes at the origin. If \( h^*(0) \) is not zero, then \( h[h^* - h^*(0)] \) is considerably smaller than \( hh^*(0) \) near the origin. The same is true of products between \( h \) and any function \( g^* \) which is the analytic extension in the unit circle of the complex conjugate of a function \( g \). Replacing \( hg^* \) by \( hg^*(0) \) is an operation which can be performed on the unit circle because \( g^*(0) \) is the zeroth-order Fourier coefficient of \( g^* \). A systematic implementation of these operations for complicated expressions such as those defining (2.9) is facilitated by the following recipe. We consider the product rule for functions on the unit circle defined by

\[
h \cdot g = PhPg,
\]

(3.11)

where

\[
Ph = \sum_{k \geq 0} h_k e^{ik\alpha}
\]

(3.12)

and where \( PhPg \) is the usual product of the functions \( Ph \) and \( Pg \). Because \( PhPg = P(PhPg) \) the product rule (3.11) is associative. This implies that replacing the usual product in any algebraic expression by the one defined above can be done consistently, independently of the order in which the operations are performed. Consequently, analytic expressions in \( h \) and \( h^* \) are transformed in the same expressions with \( h, h^* \) replaced by \( Ph \) and \( Ph^* \).

Turning now to the expressions (2.10) and (2.11) defining the basic equation (2.9) we evaluate them replacing usual products by the ones defined by (3.11). For instance in order to evaluate \( |h|^{-1} \) one writes first

\[
|h|^{-1} = h^{-1/2}(h^*)^{-1/2}
\]

*There seems to be a close relation between this procedure and the ones used in the vortex sheet problems [30, 31]. and then replaces the usual product. Thus

\[
|h|^{-1} \text{ becomes } (hh^*_0)^{-1/2},
\]

where \( h_0 \) is the zeroth-order Fourier coefficient of \( h \) and \( Ph = h \). In the example above we assumed that \( h \) is an invertible function and used the fact that the \(-1/2\) powers of \( h, h^* \) are then defined by convergent power series.

Simple computations yield

\[
F(h) \text{ becomes } \mathcal{F}(h) = 2(h^*_0)^{-1} - |h_0|^{-2}(I + D)h
\]

(3.13)

and

\[
G(h) \text{ becomes } \mathcal{G}(h) = 2(h^*_0)^{-3/2}D(I - D^2)h^{-1/2}
\]

(3.14)

for invertible functions \( h \) satisfying \( Ph = h \). Eq. (2.9) becomes the corresponding equation with \( F(h), G(h) \) replaced according to (3.13), (3.14):

\[
\frac{\partial h}{\partial t} = \mathcal{F}(h) + \tau \mathcal{G}(h).
\]

(3.15)

The equation for \( h_0 \), the zeroth-order Fourier coefficient of the solution of (3.15) decouples easily:

\[
\frac{\partial h_0}{\partial t} = (h^*_0)^{-1}.
\]

(3.16)

The solution of (3.16) is the very same \( r(t) \) given in (3.1), (3.2), that is, the zeroth-order Fourier coefficient of solutions of (3.15) is the exact solution of the original equation (2.9) corresponding to a circular initial interface. Performing the same change of dependent variable (3.3) and time scale (3.4) as before, eq. (3.15) becomes

\[
\frac{\partial g}{\partial s} = 2 - 2g - Dg + 2\tau e^{-s} D(I - D^2)g^{-1/2}.
\]

(3.17)
We used the same convention \( r(0) = 1 \) as before. Note that the new equation is local (i.e. differential). The constant function \( g = 1 \) is a steady solution of (3.17) and the equation for infinitesimal disturbances around it is identical to the corresponding equation for the original problem, (3.7). Thus (3.17) agrees at least to first order with the original equation.

4. Uniform existence of solutions

In the previous section we derived a differential equation (3.17) which holds for functions \( g \) on the unit circle which can be extended to analytic functions in the exterior of a neighborhood of the origin. Clearly, the same equation holds for the extended functions (denoted again \( g \)). In order to study functions \( g \) which are close to \( g = 1 \) we perform a change of dependent variable \( g \to 1 + g \). In order to have functions defined in a neighborhood of the origin rather than a neighborhood of infinity we perform an inversion of the independent variable \( w \to 1/w \). Finally, in order to investigate the \( \tau \) small behavior we perform a time-dependent dilation of the independent variable, \( w \to e^{\tau} w \). Therefore the new independent variable \( z \) is given by

\[
z = w^{-1} e^{\tau}
\]

and the new dependent variable \( u \) is given by

\[
g(w,s) = 1 + u(w^{-1} e^{\tau}, s).
\]

The function \( u \to (1 + u)^{-1/2} \) has the expansion

\[
(1 + u)^{-1/2} = 1 - \frac{1}{2} u + \sum_{j=2}^{\infty} c_j u^j.
\]

We set

\[
f(u) = \sum_{j=2}^{\infty} c_j u^j.
\]

Because \( D \) is dilation invariant and changes sign under inversions, eq. (3.17) becomes

\[
\frac{\partial u}{\partial s} + 2u + \tau e^{-\tau} D(D^2 - I) u \\
= 2\tau e^{-\tau} D(D^2 - I) f(u)
\]

with initial datum

\[
u(z,0) = g(z^{-1}, 0) - 1 = u_0(z).
\]

We will prove that a mixed Newton/backward Euler scheme of the form

\[
L_0(u^{(n)}) = 2\tau e^{-\tau} D(D^2 - I) \\
x \left[ f(u^{(n-1)}) + f'(u^{(n-1)})(u^{(n)} - u^{(n-1)}) \right],
\]

\[
u^{(n)}(z, 0) = u_0(z)
\]

converges in an appropriate sense and the limit is the solution of (4.5). In (4.7) we denoted by \( L_0 \) the linear operator on the left-hand side of (4.5). The differences

\[
u^{(n)}(z) - u^{(n-1)}(z)
\]

\((n \geq 1)\) will satisfy equations of the form

\[
L_b v = R
\]

where

\[
L_b u = \frac{\partial u}{\partial s} + 2v + \tau e^{-\tau} D(D^2 - I) (v + bv),
\]

the functions \( b \) are determined inductively by

\[
b^{(n)} = -2f'(u^{(n-1)})
\]

\((n \geq 1)\) and \( R \) is given by

\[
R = 2\tau e^{-\tau} D(D^2 - I) \\
x \left[ f(u^{(n-1)}) - f(u^{(n-2)}) - f'(u^{(n-2)})(u^{(n-1)} - u^{(n-2)}) \right]
\]

\((4.13)\)
for $n \geq 2$, and by

$$R = 2\tau e^{-1} D(D^2 - 1) \left[ f(u^{(0)}) - \frac{1}{2} u^{(0)} \right] - 2u^{(0)}$$  \hspace{1cm} (4.14)$$

for $n = 1$. The initial condition for (4.10) is, of course,

$$v(z,0) = 0$$  \hspace{1cm} (4.15)$$

and the initial guess $u^{(0)}$ can be taken to be

$$u^{(0)}(z,s) = u_0(z).$$  \hspace{1cm} (4.16)$$

The perturbation $L_0 - L_0$ of $L_0$ is of the same differential order as $L_0$ and has coefficients which are computed in the iteration. This presents the danger of a loss of domain of analyticity of the order of $\|b\|$ at each step of the iteration. This would be a catastrophic loss because the size of $b$ is small but not evanescent (does not diminish to zero during the iteration) and therefore one would run out of domain of analyticity in a finite number of steps. In reality this does not happen if one chooses the norms carefully.

For $\rho > 0$ consider the class of analytic functions

$$B_\rho = \left\{ v \mid v = \sum_{j=0}^{\infty} v_j z^j, \|v\|_\rho < \infty \right\},$$  \hspace{1cm} (4.17)$$

with

$$\|v\|_\rho = \sum_{j=0}^{\infty} |v_j| \rho^j.$$  \hspace{1cm} (4.18)$$

Similarly, for $\rho > 0$ and $S > 0$ consider the class of functions

$$B_{\rho,S} = \left\{ v \mid v = \sum_{j=0}^{\infty} v_j(s) z^j, 0 \leq s \leq S, \|v\|_{\rho,S} < \infty \right\}$$  \hspace{1cm} (4.19)$$

where

$$\|v\|_{\rho,S} = \sum_{j=0}^{\infty} \sup_{0 \leq s \leq S} |v_j(s)| \rho^j.$$  \hspace{1cm} (4.20)$$

The spaces $B_{\rho, B_{\rho,S}}$ are Banach algebras, i.e. all Cauchy sequences are convergent and the norms are submultiplicative. Any analytic function

$$f(u) = \sum_{j=0}^{\infty} f_j u^j$$

can be viewed as a map between these spaces. If $f, f', f''$ belong to some $B_\rho$, then two things happen. First $f$ is an analytic change of variables in the ball of radius $r$ in all the spaces $B_{\rho,S}$ and

$$\|f(u)\|_{\rho,S} \leq \|f\|,$$  \hspace{1cm} (4.21)$$

for all $u$ satisfying $\|u\|_{\rho,S} \leq r$. Secondly, the Taylor expansion inequalities

$$\|f(u_2) - f(u_1)\|_{\rho,S} \leq \|f''\| \|u_2 - u_1\|_{\rho,S}$$  \hspace{1cm} (4.22a)$$

and

$$\|f(u_2) - f(u_1) - f'(u_1)(u_2 - u_1)\|_{\rho,S} \leq \frac{1}{2} \|f''\| \|u_1 - u_2\|_{\rho,S}^2$$  \hspace{1cm} (4.22b)$$

hold for all $\rho, S$ if $\|u_1\| \leq r, \|u_2\| \leq r$. (In this section the letter $r$ will be used to designate a small parameter and not the circular solution $r(t)$ of the previous sections.)

Eq. (4.10) can be solved in the spaces $B_{\rho,S}$ with no loss. For later convenience we write the right-hand side $R$ of (4.10) in the form

$$R = g + \tau e^{-1} D(D^2 - 1) f.$$  \hspace{1cm} (4.23)$$

Lemma. Let $\rho, S$ be arbitrary. Assume $b \in B_{\rho,S}$ has norm less than one. Assume $f \in B_{\rho,S}, g \in B_{\rho,S}$ are arbitrary. Then the solution $v$ of

$$L_\rho v = R,$$

$$v(z,0) = 0$$
with $R$ defined in (4.23) and $L_b$ in (4.11) satisfies the inequality

$$||v||_{p, s} \leq (||f||_{p, s} + \frac{1}{2}||g||_{p, s})(1 - ||b||_{p, s})^{-1}.$$  

Proof. The $j$th power series coefficient of $v, v_j(s)$, satisfies the integral equation

$$v_j(s) = \int_0^s E_{j,r}(s, \sigma)$$

$$\times [R_j(\sigma) - \tau j(j^2 - 1) e^{-\sigma} (bv)_j(\sigma)] d\sigma$$

with

$$E_{j,r}(s, \sigma) = \exp[-2(s - \sigma)$$

$$- \tau j(j^2 - 1)(e^{-\sigma} - e^{-s})],$$

$$R_j(\sigma) = g_j(\sigma) + \tau j(j^2 - 1)e^{-\sigma} f_j(\sigma),$$

and

$$(bv)_j(\sigma) = \sum_{k=0}^j b_{j-k}(\sigma) v_k(\sigma).$$

The elementary solution $E_{j,r}$ satisfies the uniform estimates

$$\int_0^s E_{j,r}(s, \sigma) d\sigma \leq \frac{1}{2}$$

and

$$\int_0^1 E_{j,r}(s, \sigma) e^{-\sigma} \tau j(j^2 - 1) d\sigma \leq 1.$$

Using these inequalities it follows that

$$\text{sup}_{0 \leq s \leq S} |v_j(s)| \leq H_j(S)$$

$$+ \sum_{k=0}^j \text{sup}_{0 \leq s \leq S} |b_{j-k}(s)| \text{sup}_{0 \leq s \leq S} |v_k(s)|$$

with

$$H_j(S) = \text{sup}_{0 \leq s \leq S} \left[ |f_j(s)| + \frac{1}{2} |g_j(s)| \right].$$

One concludes by multiplying the above inequality by $\rho^j$ and summing in $j$.

We return now to the scheme (4.7). Because the function $f$ defined in (4.4) is analytic in a neighborhood of the origin and $f'(0) = 0$, it follows that we can find $r > 0$ such that

$$||f'||_r \leq \frac{1}{4}.$$  

(4.24)

Take arbitrary $\rho > 0$, $S > 0$. If

$$||u^{(n-1)}||_{p, s} \leq r$$  

(4.25)

then it follows from the definition (4.12) of $b^{(n)}$ and (4.21) applied to $f'$ that

$$||b^{(n)}||_{p, s} \leq \frac{1}{2}.$$  

(4.26)

One can apply then the lemma and conclude, using (4.22b) that

$$||v^{(n)}||_{p, s} \leq 2||f'||_r ||v^{(n-1)}||^2_{p, s}$$

if $n \geq 2$ or

$$||v^{(1)}||_{p, s} \leq 5||u_0||_p$$

if $n = 1$. We prove therefore by induction

Theorem 1. There exist two numbers $\epsilon$ and $C$ such that, if the initial datum $u_0$ satisfies

$$||u_0||_p \leq \epsilon$$

for some $\rho > 0$ then the solution $u(z, s)$ of (3.5), (3.6) exists for all $s$, belongs to $B_{p, S}$ for all $S > 0$ and satisfies

$$||u||_{p, s} \leq C||u_0||_p.$$ 

Note that $\epsilon, C$ are absolute constants, independent of both the radius of analyticity $\rho$ and of $\tau$.  

Let us consider the difference between a solution $u$ of (4.5), (4.6) and the exact solution

$$U(z, s) = e^{-2s} u_0(z)$$

(4.29)

with the same initial data of the $\tau = 0$ problem. It satisfies

$$L_0(u - U) = \tau e^{-s} D(D^2 - I)
\times \left[2[f(u) - f(U)] + 2f(U) - U\right].$$

(4.30)

Applying the lemma with $b = 0$ and (4.22a) we deduce

$$\|u - U\|_{\rho, s} \leq 2\|f(U) - U\|_{\rho, s}
+ 2\|f'(U)\|_{\rho} \|u - U\|_{\rho, s}.$$ 

But, $r$ is such that (4.24) holds, so we obtain

$$\|u - U\|_{\rho, s} \leq 3\|u_0\|_{\rho}.$$ 

(4.31)

The right-hand side of (4.31) does not vanish when $\tau$ does but has optimal dependence on $\rho$. In order to get optimal dependence on $\tau$ we write

$$(u - U)(\cdot, s)$$

$$= \int_{0}^{\tau} e^{-2(s-\sigma)} \tau D(D^2 - I)
\times \left[2[f(u(\cdot, \sigma)) - u(\cdot, \sigma)]\right] d\sigma,$$

(4.32)

apply theorem 1 and the Cauchy inequalities and deduce

$$\|u - U\|_{\rho} \leq C_1 (1 - \rho'/\rho)^{-4} \tau e^{-\delta} \|u_0\|_{\rho}$$

(4.33)

for any $\rho' < \rho$. The constant $C_1$ is an absolute constant.

Returning now to the original variables and eq. (3.17) we obtain

**Theorem 2.** There exist absolute constants $\varepsilon > 0$, $C > 0$ such that, if the initial datum

$$g_0(w) = 1 + \sum_{j=0}^{\infty} g_j(0) w^{-j}$$

satisfies

$$\sum_{j=0}^{\infty} |g_j(0)| \rho' \leq \varepsilon$$

for some $\rho > 0$, then, for any $\tau \geq 0$ there exists a unique global solution

$$g^{(\tau)}(w, s) = 1 + \sum_{j=0}^{\infty} g_j(s) w^{-j}$$

of eq. (3.17) defined for $s \geq 0$, $|w| > \rho^{-1} e^s$. This solution satisfies

$$\sum_{j=0}^{\infty} |g_j(s)| (\rho e^{-\delta})^{-j} \leq C\varepsilon.$$ 

The solution corresponding to $\tau = 0$ is explicit

$$g^{(0)}(w, s) = 1 + e^{-2s} \sum_{j=0}^{\infty} g_j(0) e^{\delta} w^{-j} \quad \text{and the difference } g^{(\tau)} - g^{(0)}$$

satisfies

$$\sup_{\rho' |w| \geq e^\delta} \left|g^{(\tau)}(w, s) - g^{(0)}(w, s)\right|$$

$$\leq \varepsilon \min\left\{C \delta (1 - \rho'/\rho)^{-4}, 3e^\delta\right\}$$

for any $\rho' \leq \rho$.

From this theorem it follows that suitable small initial perturbations of the circular interface give rise to solutions for $\tau \geq 0$. The difference between these solutions and the corresponding $\tau = 0$ solutions remains small at the moment when singularities of the latter reach the unit circle.

**5. Conclusions**

We derive a local equation which approximates the evolution equation for the conformal map...
describing a Hele-Shaw bubble. This map takes the complement of the unit disk in the \( w \) complex plane to the region occupied by viscous fluid. The local equation corresponding to the zero surface tension case (\( \tau = 0 \)) is explicitly integrable. Let \( R \) be the complement of any disk centered at the origin in the \( w \) plane. If \( R \) is free of singularities of the \( \tau = 0 \) solution until \( t = t(R) \) then \( R \) is free of singularities of small \( \tau > 0 \) solutions with corresponding initial data until \( t = t(R) \). The difference between these solutions is small on \( R \) in this time interval. Theorem 2 gives this statement in quantitative form. Nothing is said about the behavior of the \( \tau > 0 \) solutions after that time. The nature of the singularities of the \( \tau > 0 \) solutions can be quite different from that of the \( \tau = 0 \) ones. Although the derivation of the local equation requires the bubble to be nearly circular, we believe that the model equation describes qualitatively much of the behavior of the full nonlocal equation.

Acknowledgements

P.C. acknowledges the support of NSF; L.K. that of ONR.

Appendix. Derivation of the flow equations

We start with a function \( h(w, t) \)

\[
h(w, t) = \sum_{n=0}^{\infty} h_n w^n \tag{A.1}
\]

with the series converging absolutely for \( \rho |w| \geq 1 \) for some \( \rho > 1 \). Then the function \( \tilde{h} \) defined by

\[
\tilde{h}(w, t) = \sum_{j=0}^{\infty} h^*_j w^j \tag{A.2}
\]

extends analytically the complex conjugate of the restriction of \( h \) to the unit circle. Abusing notation and recalling the definition (2.7) of the operator \( A \) we see that

\[
Ah = h - \frac{1}{2} h_0, \quad \tilde{A}h = \frac{1}{2} \tilde{h}_0 \tag{A.3}
\]

and that

\[
A(h + \tilde{h}) = h \tag{A.4}
\]

if \( h_0 \) is real. To describe the fluid we use two functions, a complex velocity potential \( \phi(z, t) \) and a Riemann mapping \( f(w, t) \). The fluid velocity is

\[
u(z, t) = \left( \frac{\partial \phi(z, t)}{\partial z} \right)^* \tag{A.5}
\]

The map \( f \) is the conformal transformation of the exterior of the unit circle onto the region occupied by the viscous fluid. The derivative

\[
h = \frac{\partial f}{\partial w} \tag{A.6}
\]

is assumed to satisfy (A.1). The interface at time \( t \) is the set \( \{ z = \gamma(\theta, t) \} \) where the function \( \gamma \) is

\[
\gamma(\theta, t) = f(e^{i\theta}, t) \tag{A.7}
\]

The real part of \( \phi \) is proportional to the pressure. Because of incompressibility and Darcy's law the pressure must obey Laplace's equation; this requires \( \phi \) to be analytic. Because of the presence of a sink at infinity the function \( \phi \) must behave like \( \ln w \) at infinity. (We abuse notation and refer to \( \phi(f(w, t), t) \) as \( \phi \).) Because the pressure must be proportional to the curvature on the interface, one must have the relation

\[
\text{Re}(\phi(e^{i\theta}, t)) = \tau \text{Im} \left( \frac{\gamma_{\theta\theta}}{\gamma_{\theta}} \right) \tag{A.8}
\]

A form of \( \phi \) which is consistent with these requirements is

\[
\phi(w, t) = \ln w + \tau A \left( \frac{1}{|h|} \left[ 1 + \frac{Dh}{2h} + \frac{(Dh)^2}{2h^2} \right] \right) \tag{A.9}
\]
with \( D = w \partial / \partial w \). Because the interface moves with the fluid velocity given in (A.5) one has

\[
\gamma_i(\theta, t) = h^{-1} \frac{\partial \phi}{\partial w} + \theta_i \gamma_\theta. \tag{A.10}
\]

Dividing by \( \gamma_\theta \) and taking the imaginary part one gets

\[
\frac{f_t}{\sqrt{h}} + \frac{f_t}{\sqrt{h}} = \frac{D \phi + \overline{D \phi}}{h}. \tag{A.11}
\]

Applying \( A \), multiplying by \( h w \) and differentiating we find

\[
h_i = (D + I) \left[ hA \left( \frac{D \phi + \overline{D \phi}}{h} \right) \right] \tag{A.12}
\]

and substituting (A.9) in (A.12) we obtain (2.8)–(2.11).

References