QUASIPERIODICITY IN DISSIPATIVE SYSTEMS:
A RENORMALIZATION GROUP ANALYSIS

Mitchell J. FEIGENBAUM
Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Leo P. KADANOFF*
The Enrico Fermi and James Franck Institutes of The University of Chicago, 5630-5640 South Ellis Avenue, Chicago, Illinois 60637, USA

Scott J. SHENKER*
The James Franck Institute of The University of Chicago, Chicago, Illinois 60637, USA

Received 29 March 1982
Revised 1 June 1982

Dynamical systems with quasiperiodic behavior, i.e., two incommensurate frequencies, may be studied via discrete maps which show smooth continuous invariant curves with irrational winding number. In this paper these curves are followed using renormalization group techniques which are applied to a one-dimensional system (circle) and also to an area-contracting map of an annulus. Two fixed points are found representing different types of universal behavior: a trivial fixed point for smooth motion and a nontrivial fixed point. The latter represents the incipient breakup of a quasiperiodic motion with frequency ratio the golden mean into a more chaotic flow. Fixed point functions are determined numerically and via an ε-expansion and eigenvalues are calculated.

1. Introduction

In recent years there has been much interest in the transition to chaotic behavior in dynamical systems. Experimental results, on fluid systems for example, suggest the existence of several distinct "routes to chaos" [1]. Significant progress has been made on the period-doubling [2–3] and intermittency [4–5] routes by focusing on low-dimensional attractors and studying their dynamics through the use of Poincaré return maps. Renormalization group analysis of these maps has provided important insight into the scaling behavior at the onset of chaos.

It is our purpose here to apply this paradigm to another "route to chaos," that of quasiperiodic behavior (two incommensurate frequencies) followed by broadband noise. While this general scenario is commonly observed experimentally, no comparison can yet be made between experiment [6] and the theory presented here.

Consider a dissipative dynamical system depending on some parameter μ having an attracting stationary solution for μ = 0. As μ is varied, one commonly observed sequence of bifurcations is a Hopf bifurcation to a periodic solution followed by a secondary Hopf bifurcation to a quasiperiodic solution. Immediately after this second bifurcation, the system's dynamics are described by a diffeomorphic (differentiable with differentiable inverse) return map $T_\mu$ of the plane [7]

\[
\begin{bmatrix}
\theta_{i+1} \\
\eta_{i+1}
\end{bmatrix} = T_\mu \begin{bmatrix}
\theta_i \\
\eta_i
\end{bmatrix}.
\]

Identifying $\theta$ and $\theta + 1$, the periodicity in $\theta$ is
expressed by introducing the map \( S \),
\[
S[\begin{bmatrix} \theta \\ r \end{bmatrix}] = \begin{bmatrix} \theta - 1 \\ r \end{bmatrix}.
\] (1.2)
and demanding that \( S \) and \( T_\mu \) commute. Close
to the second bifurcation there will be an
attracting invariant curve in the plane topologically
equivalent to a circle. As long as this
invariant circle exists, the dynamics of the
associated attractor must be smooth; i.e., there
can be no broadband noise. However, the
pioneering numerical studies of Curry and
Yorke [8] on diffeomorphisms of the plane sug-
gest that as \( \mu \) is increased further the invariant
circle breaks up and chaotic behavior appears
soon thereafter. Thus, the emergence of chaos
after quasiperiodic motion is linked to the de-
struction of invariant circles.

Each attracting invariant circle is charac-
terized by a winding number \( W \),
\[
W = \lim_{i \to \infty} \frac{\theta_i - \theta_0}{i},
\] (1.3)
where \([\theta_0] \) can be any point in the basin of
attraction. The winding number represents the
ratio of the two frequencies in the quasiperiodic
regime. When the frequencies are commen-
surate then \( W \) is rational, say \( W = P/Q \), with \( P \)
and \( Q \) relatively prime integers. In the typical
case considered here, the attractor will be a
stable \( Q \)-cycle; i.e., a set of points \([\theta_n] \) such that
\[
T[\begin{bmatrix} \theta_n \\ r_n \end{bmatrix}] = \begin{bmatrix} \theta_{n+1} \\ r_{n+1} \end{bmatrix}
\] (1.4)
and
\[
\theta_n = \theta_0 + P, \quad r_n = r_0.
\] (1.5)
One could also view these as \( Q \) distinct attrac-
ting fixed points of the map \( T^Q \). Associated
with this stable \( Q \)-cycle will be an unstable
\( Q \)-cycle whose unstable manifold flows
smoothly into the stable cycle, forming the in-
variant circle. Aronson et al. [9] have recently
given a beautifully detailed description of the
disappearance of invariant circles with rational
winding numbers.

When the frequencies are incommensurate \( W \)
is irrational and the attractor is the entire in-
vARIANT curve. We ask the question how does an
invariant circle with irrational winding number
break up? This is the dissipative analogue of the
subject of an earlier series of papers exploring
the disappearance of KAM curves in the area-
preserving "standard map" [10-14]. As in the
previous work, we choose to study a particular
irrational winding number, the reciprocal of the
golden mean,
\[
\tilde{W} = \frac{\sqrt{5} - 1}{2},
\] (1.6)
because of its extremely simple continued fraction expansion [15]
\[
\tilde{W} = \langle 1111 \ldots \rangle = \\
\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\] (1.7)
Using a method introduced by J. Greene [10] we
study this irrational winding number by con-
sidering a series of rational winding numbers
\( W_n = P_n/Q_n \) that coverate to \( \tilde{W} \)
\[
\lim_{n \to \infty} W_n = \tilde{W}.
\] (1.8)
We assume, following Greene, that the prop-
ties of the stable \( Q_n \) cycles with winding number
\( W_n \) will, in the limit of large \( n \), accurately
represent the properties of the invariant curve
with winding number \( \tilde{W} \). It turns out that the
optimal choice for the \( W_n \) are successive trun-
cations to the continued fraction expansion
which, in the case of \( \tilde{W} \), yields

\[
W_n = F_n/F_{n-1},
\]

where \( F_n \) is the \( n \)th Fibonacci number. The Fibonacci numbers satisfy the recursion relation

\[
F_{n+1} = F_n - F_{n-1}
\]

and have initial conditions

\[
F_0 = 0, \quad F_1 = 1.
\]

Thus, our study of invariant curves reduces to the study of the fixed points of the sequence of planar maps

\[
T^{(n)} = T^{F_{n-1}}S^{F_n},
\]

which obey the recursion relations

\[
\begin{align}
T^{(n+1)} &= T^{(n)} \circ T^{(n-1)}, \\
T^{(n+1)} &= T^{(n)} \circ T^{(n)}.
\end{align}
\]

The particular map we study is

\[
T = \begin{bmatrix} \theta + \Omega - b r - (K/2\pi) \sin 2\pi \theta \\ b r - (K/2\pi) \sin 2\pi \theta \end{bmatrix},
\]

which has constant Jacobian \( b \). For the area-preserving case, \( b = 1 \), this is merely the standard map and in the singular limit \( b = 0 \) it reduces to the one-dimensional map on a circle,

\[
f(\theta) = \theta + \Omega - K/2\pi \sin 2\pi \theta.
\]

The circle map is a diffeomorphism for \( |K| < 1 \) and acquires a cubic inflection point at \( |K| = 1 \). Intuitively, one might expect the longtime behavior of the planar map with \( 0 < b < 1 \) will, for many purposes, be modelled by the \( b = 0 \) limit. Following that reasoning, one of us has recently investigated the analogous quasi-periodic phenomena in the circle map [16]. Two interesting classes of scaling behavior were found, one for \( |K| < 1 \) and the other for \( |K| = 1 \). Denoting by \( \Omega \) the value of \( \Omega \) such that there is a cycle with winding number \( W_n \) passing through \( \theta = 0 \) (this will, of course, depend on \( K \)) and letting \( \Omega_\infty \) denote the accumulation point of that sequence, it was found that for both cases

\[
\Omega_n - \Omega_\infty \approx \delta^n.
\]

Furthermore, the distance from \( \theta = 0 \) to the nearest point on the cycle, which is given by \( f^F(0) - F_{n-1} \), also converged geometrically;

\[
f^F(0) - F_{n-1} = \alpha^{-n}
\]

with \( \Omega = \Omega_n \). When \( |K| < 1 \), \( \alpha \) and \( \delta \) took on their "trivial" values

\[
\alpha = -\tilde{W}^{-1} = -1.6180339\ldots, \\
\delta = -\tilde{W}^{-2} = -2.6180339\ldots,
\]

whereas for \( |K| = 1 \)

\[
\alpha = -1.28857 \pm 0.0002, \\
\delta = -2.83360 \pm 0.0003.
\]

Similar to eq. (1.12) define the map \( f^{(n)} \) by

\[
f^{(n)}(\theta) = f^{F_{n-1}}(\theta) - F_n
\]

and note that the \( f^{(n)} \) obey two recursion relations identical to (1.13)

\[
\begin{align}
f^{(n+1)} &= f^{(n)} \circ f^{(n-1)}, \\
f^{(n+1)} &= f^{(n-1)} \circ f^{(n)}.
\end{align}
\]

At \( \Omega = \Omega_\infty \) and \( n \) large, it was found that for \( |\theta| \ll 1 \)

\[
f^{(n)}(\theta) \approx \alpha^{-n}f(\alpha^*\theta).
\]

For \( |K| < 1 \), \( f \) is merely the linear map

\[
f(x) = -1 + x.
\]
For $K = 1$, however, $\tilde{f}$ is a universal and non-trivial function. Near the origin it is apparently an analytic function of $\theta^3$.

Combining eq. (1.21) with (1.22) we find that the function $\tilde{f}$ must satisfy two equations,

\[ \tilde{f}(x) = \alpha \tilde{f}(\alpha^{-2}x), \]
\[ \tilde{f}(x) = \alpha^2 \tilde{f}(\alpha^{-1}x). \]

Ref. 16 provided strong numerical evidence for the existence of the function $\tilde{f}$ in eq. (1.22) but did not investigate directly the equations (1.24). Our aim here is to analyze these equations using a renormalization group framework akin to that used to study period doubling [2].

In section 2 we formulate this renormalization group framework, setting up eqs. (1.24) as fixed point equations. In sections 3 and 4 we find and analyze solutions to (1.24) corresponding to $|K| < 1$ and $K = 1$ respectively. We return to the original subject of this paper in section 5 where we study the incipient break-up of invariant circles in the map (1.14). We find that for the values of $K$ and $\Omega$ where the invariant circle with winding number $\tilde{W}$ breaks up, the maps $T^{(n)}$ of eq. (1.13) exhibit a scaling behavior

\[ T^{(n)} = L^{-n} \tilde{T} L^n, \]

where

\[ L \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} \alpha x \\ \alpha^2 y \end{array} \right] \]

and $\tilde{T}$ is the function

\[ \tilde{T} \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} \tilde{f}(x^3 + y^3) \\ 0 \end{array} \right]. \]

Beyond this fixed point, we believe that chaotic motion can occur. Thus, we argue that one way to drive a transition from quasiperiodic to chaotic behavior is to have the motion of the circle become singular in the sense of the nontrivial fixed point of equations (1.24).

Many of the results of the present paper were obtained in parallel by D. Rand, S. Ostlund, J. Sethra, and E. Siggia, with whom we have had frequent and fruitful exchanges [17]. Tsuda [18] has discussed piecewise continuous solutions of (1.24).

2. Formulation of renormalization group

Choosing a value for $\alpha$, we define a sequence of functions $f_n$,

\[ f_n(x) = \alpha^nf^{(n)}(\alpha^{-n}x). \]

These functions obey the recursion relations

\[ f_{n+1}(x) = \alpha f_n(\alpha f_{n-1}(\alpha^{-2}x)), \]
\[ f_{n+1}(x) = \alpha^2 f_{n-1}(\alpha^{-1}f_n(\alpha^{-1}x)), \]

which are similar in spirit to the period-doubling recursion relation

\[ g_{n+1}(x) = \alpha g_n(g_n(\alpha^{-1}x)) \]

found in ref. 2. There are two crucial differences between the two recursion schemes, both related to the second-order nature of the recursion relation (1.10) of the Fibonacci numbers. Here, one must specify both $f_n$ and $f_{n-1}$ in order to produce $f_{n+1}$. If $F$ is the space of functions we are considering, then our recursion relations (2.2) are really transformations on $F \times F$ to $F$. To properly formulate a renormalization group transformation we must enlarge this to a transformation taking $F \times F$ back to itself. One way to do this is to consider transformations that map $[f_n, \tilde{f}_n]$ onto $[f^{(n)}, \tilde{f}^{(n)}]$, such as

\[ R_a \left[ \begin{array}{c} u(x) \\ v(x) \end{array} \right] = \left[ \alpha u(\alpha v(\alpha^{-2}x)) u(x) \right], \]
\[ R_b \left[ \begin{array}{c} u(x) \\ v(x) \end{array} \right] = \left[ \alpha^2 v(\alpha^{-1}u(\alpha^{-1}x)) u(x) \right]. \]
Eqs. (2.4) will clearly have the same fixed point structure as (2.2).

The second difference is that there are two distinct and inequivalent recursion relations (2.2), each sufficient to define $f_{n+1}$ in terms of $f_n$ and $f_{n-1}$. However, if we choose as our initial conditions:

$$f_0(\alpha^{-1}f(x)) = \alpha^{-1}f_1(\alpha f(x))$$  

(2.5)

then the two recursion relations become equivalent.

Direct iteration of the function $f$ (eq. (1.15)), which automatically ensures (2.5), indicates that for certain values of $\alpha$ and $\Omega$ this series of functions reaches a limiting fixed point function $\bar{f}(x)$, which then clearly satisfies eqs. (1.24) [16]. We would like to know how initially small deviations about this fixed point $\bar{f}$ increase or decrease upon iterating (2.2). Set

$$f_n(x) = \bar{f}(x) + \epsilon h_n(x).$$  

(2.5)

Then eqs. (2.2) demand that to first order in $\epsilon$,

$$h_{n+1}(x) = \alpha h_n(\alpha f(\alpha^{-1}x)) + \alpha^2 \bar{f}(\alpha^{-1}f(\alpha^{-1}x))h_n(\alpha^{-1}x).$$  

(2.6a)

$$h_{n-1}(x) = \alpha^2 h_n(\alpha^{-1}f(\alpha^{-1}x)) + \alpha h_n(\alpha^{-1}f(\alpha^{-1}x))h_n(\alpha^{-1}x).$$  

(2.6b)

There are two kinds of solutions to these equations. Those in the commuting subspace which are at once solutions to both of these equations, and those in the noncommuting subspace which are solutions to only one of the two equations. We must emphasize this seemingly obvious distinction because we can, in practice, only analyze one of the two equations at a time, and must use the other equation as a guide to which solutions to keep.

We can also investigate deviations about the fixed point using the renormalization group transformation $R_\mu$ ($\mu = a$ or $b$). Let $D_{R_\mu}$ denote the linearization of $R_\mu$ about the fixed point $[\bar{f}]$.

Then, rewriting (2.6)

$$D_{R_a}\left[\begin{array}{c} u(x) \\ v(x) \end{array}\right] = \left[\begin{array}{c} \alpha u(\alpha^{-1}f(\alpha^{-1}x)) + \alpha^2 \bar{f}(\alpha^{-1}f(\alpha^{-1}x))v(\alpha^{-1}x) \\ u(x) \end{array}\right].$$  

(2.7a)

$$D_{R_b}\left[\begin{array}{c} u(x) \\ v(x) \end{array}\right] = \left[\begin{array}{c} \alpha^2 v(\alpha^{-1}f(\alpha^{-1}x)) + \alpha \bar{f}(\alpha^{-1}f(\alpha^{-1}x))u(\alpha^{-1}x) \\ u(x) \end{array}\right].$$  

(2.7b)

Note that

$$D_{R_\mu}\left[\begin{array}{c} h_n \\ h_{n-1} \end{array}\right] = \left[\begin{array}{c} h_{n+1} \\ h_n \end{array}\right].$$  

(2.8)

Thus, the renormalization group transformation $R_\mu$ accurately embodies all important facets of the recursion relations (2.2), having exactly the same fixed point structure and linearization equations.

Finding an eigenfunction of eqs. (2.7) is equivalent to searching for an initial deviation $h_\delta(x)$ such that $h_n(x) = \lambda^n h_\delta(x)$. Borrowing a technique from ref. 2, we find a family of such solutions generated by coordinate transformations. Consider the sequence of functions $f_n(x)$ defined by

$$f_n(x) = H_n \circ \bar{f} \circ H_n^{-1}(x).$$  

(2.9)

where $H_n(x)$ is close to the identity

$$H_n(x) = x + \epsilon \sigma_n(x).$$  

(2.10)

We demand that

$$H_n^{-1}(x) = \alpha H_n(\alpha^{-1}x),$$  

(2.11)

or, equivalently,

$$\sigma_{n+1}(x) = \alpha \sigma_n(\alpha^{-1}x).$$  

(2.12)

which then implies that $f_n$ satisfy the recursion
relations (2.2). Expanding to first order in $\epsilon$, the functions then become

$$f_n(x) = \tilde{f}(x) + \epsilon h_n(x),$$

(2.13)

with

$$h_n(x) = \sigma_n(\tilde{f}(x)) - \tilde{f}'(x)\sigma_n(x).$$

(2.14)

It is then clear that any sequence of functions $\sigma_n(x)$ satisfying eq. (2.12) and such that

$$\sigma_{n+1}(x) = \lambda \sigma_n(x)$$

(2.15)

produces an eigenfunction with eigenvalue $\lambda$. Eqs. (2.15) and (2.12) taken together imply that

$$\alpha \sigma_0(\alpha^{-1}x) = \lambda \sigma_0(x)$$

(2.16)

The smooth solutions to eq. (2.16) are simple monomials

$$\sigma_0(x) = x^\alpha, \quad \lambda = \alpha^{1-p},$$

(2.17)

with $p \gg 1$.

The only relevant eigenvalue, $\lambda = \alpha$, corresponds to a translation while the marginal eigenvalue, $\lambda = 1$, corresponds to a magnification. There will, of course, be other eigenfunctions besides those generated by coordinate changes; those must be found by directly solving eqs. (2.6).

3. Trivial solutions and perturbations

3.1. Linear solution

One obvious solution to eqs. (1.24), which corresponds to $|K| < 1$, is

$$\tilde{f}(x) = -1 + x,$$

(3.1)

with $\alpha$ given by $-\tilde{W}^{-1}$. For notational clarity, this value of $\alpha$ will be denoted by $\beta$. The spectrum of $DR_\mu$ around this fixed point certainly includes those eigenfunctions generated by coordinate changes, producing eigenvalues

$$\lambda = \beta^{1-p}$$

(3.2)

with $p \gg 1$. The $p = 0$ eigenvalue is eliminated because the corresponding eigenfunction, given by eq. (2.14), is identically zero.

There are other eigenfunctions. Let $h_n(x) = \lambda^m, \Psi_m(x)$, $\mu$ taking on the value $a$ or $b$ depending on whether $h_n(x)$ solves eqs. (2.6a) or (2.6b). The resulting equations are

$$\Psi_m^a(x) = (\beta \lambda_m^{-1})\Psi_m^a(-\beta + \beta^{-1}x) + (\beta \lambda_m^{-1})^2 \Psi_m^a(\beta^{-2}x),$$

(3.3a)

$$\Psi_m^b(x) = (\beta \lambda_m^{-1})^2 \Psi_m^b(-\beta^{-1} + \beta^{-2}x) + (\beta \lambda_m^{-1}) \Psi_m^b(\beta^{-1}x).$$

(3.3b)

If we look only in the space of polynomials, and let $\Psi_m^a(x)$ have as its highest power $x^m$, then both eqs. (3.3) yield the equation

$$1 = \beta^{1-m} \lambda_m^{-1} + (\beta^{1-m} \lambda_m^{-1})^2.$$  

(3.4)

Eq. (3.4) has two solutions,

$$\lambda_m = \beta^{-m},$$

(3.5)

which corresponds to the eigenfunctions generated by coordinate changes already considered, and

$$\lambda_m = -\beta^{2-m}$$

(3.6)

which is a new solution. For $m = 0$, the new solution is

$$h_n(x) = (-\beta^2)^n,$$

(3.7)

which is certainly relevant and in the commuting space (satisfies both eqs. (3.3)). This perturbation corresponds to a change in winding number (change in $\Omega$ in eq. (1.15)) and is res-
ponsible for the geometric convergence rate of 
\( \delta = -\beta^2 \) when \( |K| < 1 \).

One can solve for the \( \Psi_m^n(x) \) with \( m \geq 1 \) but these solutions depend on \( \mu \) so they are in the noncommuting space and will be neglected.

3.2. More general solutions

In this section we shall look at functions \( f(x) \)
which have the following properties:

(a) the \( x \) derivative of these functions is always nonnegative:

(b) as \( x \to 0 \), they are analytic functions of \( x^m \).

\( f(x) = a + bx^m + \cdots , \) \hspace{1cm} (3.8)

where \( x' \) means \( x|x|^{-1} \).

(c) the unique zero of \( f(x) \) is at \( x = 1 \), and is a point of singularity of the form \((x - 1)^{\alpha} \), i.e., as \( x \to 0 \)

\( f(x) = c(x - 1)^{\alpha} + \cdots . \) \hspace{1cm} (3.9)

Therefore, the \( |K| < 1 \) case considered previously corresponds to \((\nu_0, \nu_1) = (1, 1) \) and the \( K = 1 \) case to be considered in section 4 corresponds to \((\nu_0, \nu_1) = (3, 1) \). In general we shall write the fixed point solution with these characteristic singularities as \( \tilde{f}_{\nu_0, \nu_1}(x) \) and the corresponding \( \alpha \)-values as \( \alpha(\nu_0, \nu_1) \). These solutions are related, as can be seen by considering the function

\( g(x) = H^{-1} \circ \tilde{f}_{\nu_0, \nu_1} \circ H_1(x) \). \hspace{1cm} (3.10)

where

\( H_1(x) = x' \). \hspace{1cm} (3.11)

Clearly, \( g(x) \) has singularity signature \((\lambda \nu_0, \nu_1/\lambda) \) and is a fixed point solution of eq. (1.24) with \( \alpha = (\alpha_{\nu_0, \nu_1})^{1/\lambda} \). Therefore if \( \tilde{f}_{\nu_0, \nu_1}(x) \) is uniquely defined by eqs. (1.24) and the boundary conditions (3.8) and (3.9) then

\( \tilde{f}_{\lambda \nu_0, \nu_1}(x) = [\tilde{f}_{\nu_0, \nu_1}(x')^{1/\lambda} \]

\( \alpha_{\lambda \nu_0, \nu_1} = [\alpha_{\nu_0, \nu_1}]^{1/\lambda} \)

\( \alpha_{\lambda \nu_0, \nu_1} = [\alpha_{\nu_0, \nu_1}]^{1/\lambda} \)

where \( a_{\nu_0, \nu_1} \) is merely \( \tilde{f}_{\nu_0, \nu_1}(0) \).

Let us define a standard situation in which \( \nu_0 = \nu_1 = \nu \) and call the quantities defined in this standard situation by the names \( \tilde{f}(\nu, x) = \tilde{f}_{\nu, \nu}(x) \), \( \alpha(\nu) = \alpha_{\nu, \nu} \) and \( a(\nu) = a_{\nu, \nu} \). Then, according to eq. (3.12)

\( \tilde{f}_{\nu_0, \nu_1}(x) = [\tilde{f}(\sqrt{\nu_0 \nu_1}, x^{\sqrt{\nu_0 \nu_1}})]^{\sqrt{\nu_0 \nu_1}}, \)

\( \alpha_{\nu_0, \nu_1} = [\alpha(\sqrt{\nu_0 \nu_1})]^{\sqrt{\nu_0 \nu_1}}, \)

\( a_{\nu_0, \nu_1} = [a(\sqrt{\nu_0 \nu_1})]^{\sqrt{\nu_0 \nu_1}}. \)

Furthermore, consideration of the inverse function

\( \tilde{f}(\nu^{-1}, x) = \tilde{f}^{-1}(\nu, a(\nu)x/a(\nu)) \)

yields the relations

\( \alpha(\nu) = \alpha(\nu^{-1}) \), \hspace{1cm} (3.15)

\( a(\nu) = [a(\nu^{-1})]^{-1} \).

These relations enable one to discuss solutions for \( \nu \) close to 1. Write \( \nu = \nu^* = 1 + \epsilon + \cdots \) and

\( \tilde{f}(\nu, x) = \tilde{f}(1, x) + \epsilon \tilde{h}(x) + \cdots . \)

From eqs. (3.15) it follows that for small \( \epsilon \)

\( \alpha(\nu) = \beta + O(\epsilon^2), \)

\( a(\nu) = -e^{\epsilon a_1} + O(\epsilon^3), \)

while eq. (3.14) implies that in order \( \epsilon \)

\( \tilde{h}(x) + \tilde{h}(1-x) + a_1 = 0. \)

Since \( \alpha \) has no first order corrections, eqs. (2.6) imply that \( \tilde{h}(x) \) obeys the two equations

\( \tilde{h}(x) = \beta \tilde{h}(-\beta + \beta^{-1}x) + \beta^2 \tilde{h}(\beta^{-2}x). \)

(3.20a)
\( \tilde{h}(x) = \beta^2 h(\beta^{-1} + \beta^{-2}x) + \beta h(\beta^{-1}x). \)  

(3.20b)

According to eqs. (3.8) and (3.9) \( \tilde{h}(x) \) has logarithmic singularities at \( x = 0 \) and \( x = 1; \)

\[ \tilde{h}(x) = -a_1 + x \ln|x| + \mathcal{O}(x), \text{ for } x \to 0, \]

(3.21a)

\[ \tilde{h}(x) = - (1-x) \ln(1-x) + \mathcal{O}(1-x), \text{ for } x \to 1. \]

(3.21b)

Notice that if \( \tilde{h}(x) \) obeys eq. (3.20b) and the symmetry condition (3.19) then the identity \( 1 = \beta + \beta^2 \) enables one to derive the fact that \( \tilde{h}(x) \) will automatically satisfy eq. (3.20a). Also notice that the boundary conditions (3.21) are consistent with eq. (3.19). Hence, in order to get the first order perturbation theory correct we need only solve eq. (3.20b) using boundary conditions (3.21). In the appendix we construct the solution \( \tilde{h}(x) \), demonstrating that to first order in \( \epsilon \) the eqs. (1.24) indeed have a solution.

4. \( K = 1 \) solution

4.1. Calculation of fixed point

We now want to calculate the fixed point solution of eqs. (1.24) which correspond to the \( K = 1 \) scaling found in ref. 16. First, notice that when \( K = 1 \), all functions \( f_n \) in eq. (2.1) including the limiting fixed point \( \tilde{f}(x) \) have no linear or quadratic terms in their Taylor series about \( x = 0 \). It is straightforward to show that any analytic function \( \tilde{f}(x) \) satisfying either of eqs. (1.24) and having \( f'(0) = 0 \) and \( f''(0) = 0 \) must have a Taylor series in \( x^3 \) about \( x = 0 \). Accordingly, we will limit our search to the space of smooth functions of \( x^3 \).

Conceivably, both eqs. (1.24) must be used to determine the fixed point we seek. This turns out not to be the case with eq. (1.24a) being the only necessary one. Basically, a functional equation is an infinite dimensional equation, and the initial conditions that determine a unique solution can be the specification of the solution on some interval. Such functional equations produce a global solution given its restriction to this interval. A "serious" functional equation demands self-consistency on this determining interval and so determines a unique solution. Accordingly, we must find a self-determining equation and interval. This is most easily done by considering the function

\[ g(x) = \alpha f(x), \]

(4.1)

which is monotone decreasing, since \( \tilde{f} \) is monotone increasing, and possesses a unique fixed point. Applying eqs. (1.24) we find two new fixed point equations,

\[ g(x) = \alpha g(g(x/\alpha^2)), \]

(4.2a)

\[ g(x) = \alpha^2 g(\alpha^{-2} g(x/\alpha)). \]

(4.2b)

Since these equations are all invariant under magnification, we may also specify the constraint

\[ g(0) = 1, \]

(4.3)

which then implies, by (4.2a),

\[ g(1) = \alpha^{-1}. \]

(4.4)

Thus, the fixed point lies in the interval \((0, 1)\). Substituting \( x = 1 \) in eq. (4.2a) yields the result

\[ g^2(\alpha) = \alpha^{-2}, \]

(4.5)

so that \( \alpha^{-2} \) is either the fixed point of \( g \) or a point of period two. The fact that \( g \circ g(x) \) is monotonic eliminates the possibility of a two-cycle so \( \alpha^{-2} \) must indeed be the fixed point of \( g(x) \).

Consider eq. (4.2a) for \( x \in [0, 1] \). \( g(x/\alpha^2) \) maps \([0, 1]\) onto \([\alpha^{-2}, 1]\) so that nowhere in the equation is \( g(x) \) needed for \( x \) outside of \([0, 1]\). Thus, eq. (4.2a) has a self-determining interval \([0, 1]\). We expect then that eq. (4.2a) restricted to \([0, 1]\) has an isolated monotonic solution in terms of
real analytic functions of $x^3$. That is, we expect that \( (4.2a) \) alone determines the fixed point solution $g(x)$ on $[0, 1]$. With this restriction obtained, eq. (4.2b) then determines $g(x)$ on $[-a_0^{-1}(0), 0]$ through the evaluation of right-hand side terms on the obtained solution to (4.2a). Repeated use of eqs. (4.2) finally allows a global determination of $g$. A solution for $g$, then, reduces to a solution of eq. (4.2a) on $[0, 1]$, which we now numerically obtain.

As a functional equation (4.2) is of course infinite dimensional. We seek a solution of the form

$$g(x) = 1 + \sum_{n=1}^{\infty} c_n x^{bn}. \quad (4.6)$$

Employing (4.4), we require (4.6) to satisfy

$$F[g](x) = g(1)g(x) - g(g(x[g(1)])^2) = 0 \quad (4.7)$$

for $x \in [0, 1]$. Loosely speaking, the vanishing of $F$ at an infinite number of points $x$ determines the infinite number of coefficients $c_n$. We obtain an $N$th order approximation by writing

$$g_N(x) = 1 + \sum_{n=1}^{N} c_n^{(N)} x^{bn} \quad (4.8)$$

and demanding that (4.7) be satisfied at $N$ points in $[0, 1]$. The accuracy of this approximation is then determined by

$$\epsilon_N = \sup_{x \in [0, 1]} |F[g_N](x)|. \quad (4.9)$$

where $\epsilon_N$ depends upon the choice of the $N$ points $x_i$ at which $F$ vanishes. The determination of the period-doubling fixed point [2] which utilized exactly the same method, was rather insensitive to the choice of points. In the present case, however, a careful choice is mandatory. Had we taken

$$x_i = i/N, \quad i = 1, \ldots, N, \quad (4.10)$$

then for $N > 3$ the numerical satisfaction of $F(x_i) = 0$ is difficult to obtain, and $\epsilon_N$ fails to decrease below $10^{-3}$ with increasing $N$. However, the choice

$$x_i = (i/N)^{1/3}, \quad i = 1, \ldots, N, \quad (4.11)$$

results in swift convergence of a Newton's method solution of $F(x_i) = 0$ for all $N$, with $\epsilon_N$ decreasing rapidly with $N$ $(\epsilon_N \sim 10^{-8} \text{ with } N = 12 \text{ on a machine with } 16 \text{ significant figures})$. The spectrum of $DF$ at this solution possesses a unique eigenvalue outside of the unit circle (at 8.675) and all others well inside the unit circle. We regard this as numerical evidence that (4.2a) indeed possesses an isolated fixed point of the desired character (see fig. 1). From eq. (4.4) we can determine $\alpha$ from our solution,

$$\alpha = -1.288575 \pm 0.000001, \quad (4.12)$$

which is in evident agreement with the iteration data.

Using an essentially identical procedure, we can find the fixed points having singularity structure $(\nu, 1)$ by considering polynomials in $x^\nu$ and points $x_i = (i/N)^{1/\nu}$. These fixed points, while not physically relevant, will prove helpful in identifying the nature of eigenfunctions in the next section.

### 4.2. Perturbations about the fixed points

When studying perturbations about the fixed point, it is important to determine the function

![Fig. 1. The fixed point function $g(x)$ calculated in the $N = 12$ approximation over the interval $[0, 1]$.](image-url)
space of allowed perturbations. Since the fixed point solution was obtained in the space of functions of $x^3$, we will first consider perturbations in that space. Furthermore, we will concentrate only on the spectrum of the linearized operator $DR$, (eq. (2.7a)), as the fixed point was obtained for the operator $R_a$ (eq. (2.4a)).

To solve, numerically, the infinite dimensional eigenvalue equation

$$DR_a[u(x)] = \lambda[v(x)]$$  \hspace{1cm} (4.13)

(with $DR_a$ evaluated at the fixed point), we consider $N$-dimensional approximates in the spirit of our previous considerations about such approximations to eq. (1.24a) (see also ref. 2). Namely, we replace (4.13) by the same equation but restricted to $\{x_i\} i = 1, \ldots, N$ with $u$ and $v$ polynomials of $x^3$ of degree $N$. In determining the spectrum, we can use any $N$ points in [0, 1], although the most natural choice is the same set used to compute the fixed point $g_N$. Table 1 lists the numerical results that follow from this method, with $N = 12$. Only those eigenvalues of magnitude greater than or equal to one are shown, and we now seek to identify each entry.

From the analytic considerations of section 2, we know that certain coordinate changes generate perturbations which are eigenfunctions of both $DR$ and $DR_a$. These eigenfunctions have eigenvalues $\alpha^{-p}$ with $p > 0$. However, if we consider only those eigenfunctions which are polynomials of $x^3$, then we find only the spectral elements $\alpha^{-3p}$ with $p > 0$. Thus, the entry $\alpha^3$ in table 1 results from coordinate changes due to magnifications. This marginal eigenvalue is not seen in the numerical work on cycles (ref. 16), as the periodic nature of $f(x)$ in eq. (1.15) automatically rules out magnifications. The other entries can be identified if we consider the fixed points $f_{1,1}$ and their spectra. When $\nu = 1$, we have the trivial solution. There, in addition to the marginal magnification, we had an eigenvalue of $-\beta^2$ corresponding to change in winding number, and eigenvalues of $-\beta$ and $-\beta^0$ corresponding to noncommuting eigenfunctions. As $\nu$ changes from 1 to 3, these three eigenvalues are connected continuously to the eigenvalues $\delta, -\alpha^3, \alpha^0$ in table I. We can then conclude that the eigenfunction corresponding to $\delta$ is responsible for changes in winding number. This explains the geometric convergence rate of the $\Omega_{n}$'s found in ref. 16. The other two eigenfunctions, corresponding to $-\alpha^3$ and $-\alpha^0$, are not in the commuting space and can be ignored.

We have found a unique relevant eigenvalue $\delta$ when only polynomials in $x^3$ are considered. We now want to consider the case

$$1 - K = \varepsilon > 0,$$  \hspace{1cm} (4.14)

in which case $f_0(x^3)$ has a perturbation of the form $\varepsilon x$. Since $\varepsilon x$ destroys the leading cubic behavior, such a perturbation is singular in that $f_0$ is no longer in the space of functions of $x^3$. It is easy to see that the coordinate transformation

$$H(x) = (x^3 + \varepsilon x)^{1/3}$$  \hspace{1cm} (4.15)

provides the required perturbation. Expanding (4.15)

$$H(x) = x + \frac{\varepsilon}{3x} + \cdots,$$  \hspace{1cm} (4.16)

we have $p = -1$ in (2.17), so that the relevant
eigenvalue for this fixed-point destroying perturbation is $\alpha^\omega$ again in agreement with the cycle iteration data (i.e., this result would imply $\nu = 2\lambda$ in the notation of ref. 16, which is true to numerical accuracy).

Of course, once we consider these more general perturbations we generate the entire spectrum $\alpha^{1-p}$ with $p > 0$. The only relevant perturbation, $p = 0$, corresponds to a translation. This eigenvalue is not observed in the cycle data [16] because all the cycles begin at $\theta = 0$.

5. Invariant circles in the planar map

We now return to our original problem, that of invariant circles of the map $T$ (eq. (1.14)) having winding number $\tilde{W}$. In particular, we are interested in how the structure of this curve depends on $K$. As described in the introduction, this invariant curve is seen as the limiting case of the set of fixed points of the maps $T^{(n)}$ defined in eq. (1.13). Consider $b$, $0 < b < 1$, to be given. For each value of $K$ there are a series of $\Omega$ values, $\Omega_n$, and initial points $\{r_0(n), \theta_0(n)\}$ such that

$$T^{(n)}[[\theta_0(n)], [r_0(n)]] = [[\theta_0(n)], [r_0(n)]]$$  \hspace{1cm} (5.1)

(we suppress the $K$ dependence of all these quantities). Eq. (5.1) imposes only two conditions on the three variables $r_0(n)$, $\theta_0(n)$, $\Omega_n$, leaving one variable, say $\theta_0(n)$, undetermined. This ambiguity can be removed by defining a quantity $D_n = \Gamma r M_n$, where $M_n$ is the tangent matrix of $T^{(n)}$ evaluated at $[\theta_0(n), r_0(n)]$, and then choosing the value of $\theta_0(n)$ by demanding that $D_n$ be a minimum. Note that this criterion is equivalent to demanding that cycles start at $\theta = 0$ in the circle map with $K = 1$, as was done in ref. 16.

Letting $n \to \infty$, we empirically find three cases

I: $D_n \to 0, \hspace{1cm} K = K_c$.

II: $D_n \to 0, \hspace{1cm} K = K_c$.

III: $D_n \to -\infty, \hspace{1cm} k > K_c$.

where $K_c$ depends on $b$. J. Greene, in his work on the standard map, found three analogous situations and concluded that a smooth continuous invariant curve with winding number $\tilde{W}$ existed for $0 \leq K < K_c$, this curve was no longer smooth at $K = K_c$, and was no longer continuous for $K > K_c$ (see Mather [19]). Our numerical results suggest that these conclusions apply to the dissipative case as well. With $b = 0.5$ and using eq. (5.2) to define $K_c$, it appeared that for $K < K_c$ the fixed points of $T^{(n)}$ were converging onto a smooth invariant curve as $n \to \infty$. This can be seen in fig. 2a where the fixed points of $T^{(13)}$ are shown for $K = 0.3$ ($Q_{13} = 377$, $P_{13} = 233$). The density of fixed points gives a discrete approximation to the invariant distribution curve, and for $K < K_c$ this appears to be a smoothly varying function. At $K = K_c = 0.978837778 \pm 0.000000002$, the invariant curve appears to have an infinite number of cubic kinks (the phenomena of kinks in invariant

Fig. 2. The fixed points of the map $T^{(13)}$ with $b = 0.5$ for (a) $K = 0.3$ and (b) $K = K_c = 0.978837778$. 
curves was first pointed out by Curry and Yorke \([8]\)). Fig. 2b shows the fixed points of \( T^{(a)} \) for \( K = K_c \), and it is clear that in addition to the kinks the invariant distribution as seen from the density of points on this curve has developed singularities which appear to include both zeroes and infinities.

When \( K > K_c \), the fixed points become unstable and period-double and there are numerous tangent bifurcations producing extraneous fixed-point pairs. This makes the numerical determination of \( \Omega_n, \theta_0(n), \) and \( r_0(n) \) both impractical and ambiguous. However, we feel that one can safely assume that no continuous invariant curve with winding number \( \tilde{W} \) exists for \( K > K_c \).

Similar to refs. 16 and 11, the convergence ratios \( \alpha_n \) and \( \delta_n \) can be defined,

\[
\delta = \frac{\Omega_{n+1} - \Omega_{n-2}}{\Omega_n - \Omega_{n-1}},
\]

\[
\alpha_n = \frac{\theta_{F_n-1}(n-1) - \theta_0(n-1) - F_{n-2}}{\theta_{F_n}(n) - \theta_0(n) - F_{n-1}}.
\]

In Table II we list the values of \( D_n, \alpha_n, \) and \( \delta_n \) for \( b = 0.5 \) and two values of \( K : K = 0.3 \) and \( K = 0.78837778 \). Note that for \( 0 < K < K_c \), \( \alpha_n \) and \( \delta_n \) converge to the “trivial” values given by (1.18). When \( K = K_c \), \( \alpha_n \) and \( \delta_n \) appear to converge to the “nontrivial” values of (1.19). This suggests that in the vicinity of \( [\theta_0(n)] \) the maps \( T^{(a)} \) obey a scaling law

\[
T^{(a)} = L^{-n} \tilde{T} L^n,
\]

where \( \tilde{T} \) is some universal function and

\[
L \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} \alpha x \\ \gamma y \end{array} \right].
\]

If (5.5) is correct, we then expect that the renormalization group transformations

\[
R_b^{(2)} \left[ \begin{array}{c} u(\zeta) \\ v(\zeta) \end{array} \right] = \left[ \begin{array}{c} L^2 \circ v \circ L^{-1} \circ u \circ L^{-1}(\zeta) \\ u(\zeta) \end{array} \right],
\]

will have a fixed point, namely \( \tilde{T}(\zeta) \).

It is easy to construct fixed points to \( R_b^{(2)} \)
given the results for the circle map. Corresponding to the $0 \leq K < K_c$ case, we have
\[
\mathcal{T}\left[\begin{array}{c}
x \\
y
\end{array}\right] = \left[\begin{array}{c}
-1 + x \\
0
\end{array}\right].
\] (5.8)

with $\alpha = \beta$ and $\gamma$ arbitrary. The $K = K_c$ solution is represented by the fixed point
\[
\mathcal{T}\left[\begin{array}{c}
x \\
y
\end{array}\right] = \left[\begin{array}{c}
\tilde{f}((x^2 + y)^{1/3}) \\
0
\end{array}\right].
\] (5.9)

with $\alpha$ and $\tilde{f}$ being the $K = 1$ circle map quantities and $\gamma = \alpha^1$. This fixed point is similar to the generalization of the one-dimensional period-doubling fixed point to planar maps [20]. The fixed point (5.9) will have two important relevant eigenvalues; $\delta$, corresponding to variations in $\Omega$ and $\alpha^2$ corresponding to decreasing $K$. There will of course be other relevant eigenvalues, generated by coordinate transformations, but which can be eliminated by appropriate normalizations. But perhaps the major physics of deviations from this incipient breakup will be caught by the two relevant directions of the one-dimensional map.

6. Conclusions

We have presented a theory that predicts certain scaling laws at the onset of chaotic behavior in a quasiperiodic dynamical system. While this theory applies only to the irrational frequency ratio $\tilde{W}$, it can be generalized in a straightforward way to any irrational winding number with a repeating continued fraction expansion. The crucial question is can this scenario be observed experimentally? The theory requires that the winding number be held constant as the nonlinear coupling is increased. At present most experiments have only one adjustable parameter, so the winding number cannot be controlled independently from the nonlinear coupling. Preventing any comparison with the present theory. It is our hope that in the future, experiments with two adjustable parameters will display this "route to chaos."

Acknowledgements

We are particularly indebted to D. Rand, S. Ostlund, J. Sethna, and E. Siggia for communicating their results prior to publication and freely sharing their ideas and insights. We have also had helpful discussions with A. Zisook, M. Widom, J. Greene, and R. MacKay. One of us (SJS) would like to acknowledge his NSF and McCormick Fellowships.

Appendix A

Exact first order solution

To avoid the difficulties produced by the inequivalence of eqs. (2.2a) and (2.2b), we choose initial conditions for these recursion relations which will guarantee the necessary commutation relations. As described by eq. (2.5), choose
\[
\begin{align*}
   f_0(x) &= -1 + x + \epsilon \beta P(x/\beta), \\
   f_1(x) &= -1 + x + \epsilon \beta^2 P(x/\beta^2),
\end{align*}
\] (A.1)

where $P(x + 1) = P(x)$. Equivalently, take
\[
\begin{align*}
   h_0(x) &= \beta P(x/\beta), \\
   h_1(x) &= \beta^2 P(x/\beta^2).
\end{align*}
\] (A.2)

To get a solution with $\nu_0 = \nu_1 = 1 + \epsilon + \cdots$ choose $P(z)$ to have a behavior
\[
\begin{align*}
   &P(z) = c + z \ln z + \cdots, \quad z \to 0, \\
   &P(z) = (z - \beta^{-1}) \ln|z - \beta^{-1}|, \quad z \to \beta^{-1}.
\end{align*}
\] (A.3)

One choice which will certainly do this is to
Now taking
\[ P(x/\beta) = Q\left(\frac{x}{\beta}\right) + Q\left(\frac{x-1}{\beta}\right) - Q(\beta^{-1}). \]  
(A.4)

with
\[ Q(z) = \frac{\sin 2\pi z}{2\pi} \ln \left| \frac{\sin \pi z}{\pi} \right|. \]  
(A.5)

Given any periodic function \( P(z) \) the solution to eqs. (2.6) is direct and relatively simple. Take as an Ansatz the proposed solution
\[ h_n(x) = \beta^{n+1} \sum_{y \in S_n} P\left(\frac{x-y}{\beta^{n+1}}\right). \]  
(A.6)

For \( n = 0 \) and \( n = 1 \) we have the set \( S_n \) containing but one element, 0,
\[ S_0 = S_1 = \{0\}. \]  
(A.7)

Now substitute eq. (A.6) into eq. (2.6b) to find
\[ \beta^{n+2} \sum_{y \in S_{n+1}} P\left(\frac{x-y}{\beta^{n+2}}\right) = \beta^{n+2}\left[ \sum_{\sigma \in S_n} P\left(\frac{x-\sigma y}{\beta^{n+2}}\right) \right] \]
\[ + \sum_{y \in S_{n-1}} P\left(\frac{x-\beta^2 y}{\beta^{n+2}}\right) \]  
(A.8)

Eq. (A.6) is then a solution if the set \( S_{n+1} \) is properly chosen. Here \( S_{n+1} \) contains two types of elements. If \( y \in S_n \) then \( (\beta y) \in S_{n+1} \). If \( y \in S_{n-1} \), then \( (\beta^2 y + \beta) \in S_{n+1} \). To represent this result in a convenient notation, say that the sets \( A \) and \( B \) are related by
\[ B = uA + v, \]  
(A.9)

if for each \( y \) in \( A \) there is one and only one corresponding \( y' \) in \( B \), with \( y' = uy + v \). In this notation, then, the set \( S_n \) is defined recursively via
\[ S_{n+1} = (\beta S_n) \cup (\beta^2 S_{n-1} + \beta) \]  
(A.10)

and the initial condition (A.7).

The properties of the set \( S_n \) are crucial to our analysis since if \( y \in S_n \) then \( h_n(x) \) has singularities at \( x = y \) and at \( x = y + \beta^n \). The singularities will, of course, determine the solution.

Notice that each \( S_n \) includes all the elements of \( S_{n-1} \). For this reason we can define \( S_n \) as the union of \( S_{n-1} \) and the new elements of \( S_n \) by writing
\[ S_n = S_{n-1} \cup T_n. \]  
(A.11)

The elements in \( T_n \) increase in magnitude as \( n \) increases. Hence lower \( n \) tends to give singularities for smaller \( x \)-values.

Note also that the second term on the right-hand side of eq. (A.4) may be interpreted as a \( y = 1 \) term. If we started from \( S_0 = S_1 = \{1\} \) and generated successive \( S_n \) by using recursion relation (A.10) then the new sets \( S_n \) thereby generated would be identical to the sets \( S_n \) except for one element. \( S_n \) always includes 0; \( S_n \), always includes a corresponding 1, instead of the 0. Therefore, except for an \( x \)-independent term, which we shall deal with later on, the solution to the recursion problem is
\[ h_n(x) = \beta^{n+1}\left[ \sum_{y \in S_n} Q\left(\frac{x-y}{\beta^{n+1}}\right) + \sum_{y \in S_{n-1}} Q\left(\frac{x-y}{\beta^{n+1}}\right) \right]. \]  
(A.12)

The set \( S_n \) is relatively easy to characterize. From the definitions (A.10) and (A.11) one can show that \( y \in T_n \) if and only if \( y \) can be written as
\[ y = \sum_{k=1}^{n-1} m_k \beta^k, \]  
(A.13)

with each \( m_k \) chosen to be either zero or one and to satisfy
\[ m_{n-1} = 1, \quad m_k m_{k-1} = 0, \]  
(A.14)

so that no two successive values of \( m_k \) are unity. The elements of \( S_n \) are distributed be-
between the extreme values $-\beta^n + 1$ and $-\beta^{n-1} + \beta$
for $n$ odd and the extreme values $-\beta^n + \beta$ and
$-\beta^{n-1} + 1$ for $n$ even. For each $n$ the number of
elements in $S_n$ is $F_n$, where $F_n$ is the Fibonacci
number defined by

$$F_n = \frac{(-\beta)^n - \beta^{-n} + 1}{(\beta + \beta^{-1})}.$$  \hfill (A.15)

When the set $S_n$ is arranged in order of size the
spacing between successive elements of $S_n$ is
either $-\beta$ or $\beta$. Hence roughly speaking, $S_n$
contains elements spaced uniformly between
$-\beta^{n-1}$ and $\beta^n$ with an average number of ele-
ments per unit length being

$$\rho = |\beta + \beta^{-1}|^{-1}.$$ \hfill (A.16)

In this rough representation

$$\sum_{y \in S_n} H(y) = (-1)^{n-1} \int_{\beta^{n-1}}^{\beta^n} dy p H(y).$$ \hfill (A.17)

Eq. (A.17) is useful for estimating the rate of
convergence of sums over $S_n$.

It is possible to prove one more result about $S_n$ and $S'_n$ which is crucial to the further analy-
sis, namely

$$S_n = 1 - S'_n \mod \beta^{n-1}.$$ \hfill (A.18)

That is to say that for each element $y$ of $S_n$ there is a corresponding element $y'$ of $S'_n$ with
$y = 1 - y' \mod \beta^{n-1})$. Consequently, eq. (A.12)
reduces to

$$h_n(x) = \beta^{n-1} \sum_{y \in S_n} \left[ Q\left(\frac{x - y}{\beta^{n-1}}\right) + Q\left(\frac{x + y - 1}{\beta^{n-1}}\right) \right].$$ \hfill (A.19)

The most useful way of writing this result is to
take a set

$$U_n = S_n \cup (1 - S_n)$$ \hfill (A.20)

and to write

$$h_n(x) = \beta^{n-1} \sum_{y \in U_n} Q\left(\frac{x - y}{\beta^{n-1}}\right).$$ \hfill (A.21)

Here $U_n$ is characterized by a recursion relation
identical to eq. (A.10),

$$U_{n+1} = (\beta U_n) \cup (\beta^2 U_{n-1} \setminus \beta),$$ \hfill (A.22)

$$U_0 = U_1 = \{0, 1\}.$$ \hfill (A.23)

Each of the elements (save 0) in $S_\infty$ appear
exactly twice in $U_n$. The prime in eq. (A.21) is a
reminder to calculate the sum in order of in-
creasing size of $|y|$. This reminder is necessary to
ensure that conditionally convergent sums like

$$\sum_{y \in U_n} 1 \frac{1}{y-\frac{1}{z}} = 0$$ \hfill (A.24)

converge and converge to zero.

One could analyze the properties of the sum
(A.21) quite directly. However, we are only
interested in the singularities in $h_n(x)$ generated
by the singularities in $Q(z)$, which are for small
$z$ like $z \ln|z|$. Hence as an ansatz, let us replace
the sum in (A.21) by the form which arises if we
replace $Q(z)$ by its small $z$ form, namely

$$h_n(x) = \sum_{y \in U_n} \{(x - y) \ln|x - y| + \cdots\}.$$ \hfill (A.25)

The ... includes analytic terms in $x$ which are
added to ensure that the sum converges in the
small $y$ region. Clearly, the sum does not con-
verge as it stands. In fact, it is hard to analyze
directly. Consequently, we instead analyze the
behavior of the $x$-derivative to eq. (A.24). We
try the ansatz

$$\hat{h}(x) = \sum_{y \in U_n} \{(\ln|x - y| - \ln|x_0 - y|) + A.$$ \hfill (A.26)

Because of the results (A.17) and (A.24) we are
sure that the definition (A.25) gives a convergent expression for a quantity which we hope to identify as the \( x \)-derivative of \( \hat{h}(x) \). There are two undetermined parameters in eq. (A.25), \( A \) and \( x_0 \).

We now set out to determine these parameters. By differentiating eq. (3.20b) we find

\[
\hat{h}(x) = \hat{h}\left(\frac{x}{\beta}\right) + \hat{h}\left(\frac{x-\beta}{\beta^2}\right).
\]

(A.26)

Let \( x \to 0 \), then the only term which contributes to \( \hat{h}(x) - \hat{h}(x/\beta) \) is the \( y = 0 \) term in eq. (A.25) which gives this difference as \( \ln|x| - \ln|x/\beta| = \ln|\beta| \). Hence

\[
\hat{h}(-\beta^{-1}) = \ln|\beta|.
\]

(A.27)

For this reason, we can pick the parameters in eq. (A.25) to be

\[
x_0 = -\beta^{-1}, \quad A = \ln|\beta|.
\]

(A.28)

Now substitute the result (A.25) into eq. (A.26) to obtain

\[
A = \sum_{y \in U_n} (\ln|\beta^2 x_0 - y + \beta| - |\ln|x_0 - y|) \\
= \sum_{y \in U_n} (|\ln|x - \beta y| - |\ln|\beta x_0 - \beta y|) \\
+ \sum_{y \in U_n} (|\ln|x - \beta^2 y - \beta| \\
- |\ln|\beta^2 x_0 + \beta - \beta^2 y - \beta|) \tag{A.29}
\]

Use the definiton (A.22) of \( U_n \) to rewrite the last sum as

\[
\sum_{y \in U_n} H(\beta^2 y + \beta) = \sum_{y \in U_n} H(y) - \sum_{y \in U_n} H(\beta y). \tag{A.30}
\]

After a rearrangement, we find

\[
A = \sum_{y \in U_n} \left( \ln|\beta^2 x_0 - y + \beta| - \ln|x_0 - y| \\
- \ln|\beta^2 x_0 + \beta - \beta y| + \ln|\beta x_0 - \beta y| \right). \tag{A.31}
\]

As \( x_0 \) approaches \(-\beta^{-1}\) the only term left in the sum is the \( y = 0 \) term which leaves

\[
A = \ln|\beta^2 x_0 + \beta| - \ln|x_0| - \ln|\beta x_0 - \beta| - \ln|\beta x_0| \\
= \ln|\beta| \tag{A.32}
\]

Hence, eq. (A.25) and (A.28) together provide a solution

\[
\hat{h}(x) = \sum_{y \in U_n} (|\ln|x - y| - |\beta^{-1} + y|) + \ln|\beta| \tag{A.33}
\]

Eq. (A.33) may then be integrated to yield our formula for the first correction to the trivial fixed point

\[
\hat{h}(x) = (x - 1) \ln|\beta| + \sum_{y \in U_n} ((x - y) \ln|x - y| \\
- (1 - y) \ln|1 - y| - (x - 1) \ln|\beta^{-1} + y|). \tag{A.34}
\]

Here we used \( \hat{h}(1) = 0 \) to set a constant of integration.

To verify this solution we need examine two points: Does \( \hat{h}(x) \) equally well satisfy eqs. (3.20a) and (3.20b)? Does it satisfy either equation? The second question could have a negative answer if the integration of eq. (A.26) to yield eq. (3.20b) led to a nonzero constant of integration. To check this point, we need only check to see that eq. (3.20b) is satisfied for one value of \( x \). Hence, set \( x = 1 \), and notice that the condition is \( \hat{h}(\beta^{-1}) = 0 \). A calculation of the sum in eq. (A.34) with 5186 terms and an estimate of the remainder gives \( \hat{h}(\beta^{-1}) = \times 10^{-7} \), which is consistent with zero. Thus, eq. (3.20b) is indeed satisfied. Next, eq. (3.20a) will also be satisfied if eq. (3.19) holds. This is the statement that \( \hat{h}(x) + \hat{h}(1-x) \) is \( x \) independent. However,
since the sum in eq. (A.34) remains invariant under the replacement of \( y \) by \( 1 - y \), we find
\[
\tilde{h}(x) + \tilde{h}(1-x) = -\ln|\beta|
\]
\[
- \sum_{n \in \mathbb{Z}} |y| + (1-y) \ln |1-y| - \ln |\beta^{-1} + y|,
\]
(A.35)
which then verifies eq. (3.19).

Note added in proof
M. Nauenberg (private communication) has constructed an elegant analysis which indicates when the solutions to equations (1.24a) and (1.24b) are identical.

References