Renormalization, Unstable Manifolds, and the Fractal Structure of Mode Locking

Predrag Cvitanović\(^{(a)}\)

Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

and

Mogens H. Jensen and Leo P. Kadanoff

The James Franck and Enrico Fermi Institutes, University of Chicago, Chicago, Illinois 60637

and

Itamar Procaccia\(^{(b)}\)

Department of Chemistry and The James Franck Institute, University of Chicago, Chicago, Illinois 60637

(Received 4 March 1985)

The apparent universality of the fractal dimension of the set of quasiperiodic windings at the onset of chaos in a wide class of circle maps is described by construction of a universal one-parameter family of maps which arises from the unstable manifold of the renormalization group. The manifold generates a universal “devil’s staircase” in which the dimension agrees with direct numerical calculations. Applications to experiments are discussed.

PACS numbers: 05.45.+b, 03.20.+i, 47.20.+m, 74.50.+r

In the context of the transition to chaos via quasiperiodicity, most attention has been paid to the local scaling behavior at a particular irrational winding number.\(^1\) Although universal behavior has been theoretically predicted,\(^{1b,6}\) its experimental verification has not followed, simply because minute changes in winding numbers lead to large changes in scaling behavior.\(^2\) It appears that of greater interest and experimental accessibility are those universal properties that are \textit{global} in the sense of pertaining to a range of winding numbers. Indeed, such a property has been found and reported by Jensen, Bak, and Bohr,\(^3\) and has to do with the set complementary to the “tongues”\(^4\) on which the dynamical system is mode locked. This set of unlocked or irrational windings has an onset of chaos Lebesgue measure zero, and apparent universal fractal dimension \(D\). Recent experiments on Josephson junction simulators\(^5,6\) and charge density waves\(^2\) have indicated the existence of this phenomenon and revealed results in agreement with the findings in Ref. 3.

For the simple circle map \(\theta_{n+1} = \theta_n + \Omega - (K/2\pi) \sin(2\pi \theta_n)\) this transition occurs at \(K = 1\); see Fig. 1. On the plotted intervals the winding number \(W\) is locked on a rational value as shown. The gaps between the locked states are “full” of locked states that add up to Lebesgue measure 1. The set of irrational winding numbers is the complement of the locked intervals. We calculated the dimension \(D\) of this set in a way slightly different from Ref. 3. We believe that \(D\) is the same for all ranges of gaps. Thus we can start with any pair of locked intervals \(P/Q\) and \(P'/Q'\). The length of the gap between them is denoted by \(\delta\). Next the locked interval \((P+P')/(Q+Q')\) is found, and the gaps of length \(s_1\) and \(s_2\) between the new interval and the preceding one are found. This “Farey tree”\(^8\) construction is continued until a large number of gap sizes \(s\) are found. The fractal dimension \(D\) is then estimated from the formula\(^9\) \(\sum R^D = 1\), where \(R = s/\delta\). Denoting the result from the \(n\)th Farey level by \(D_n\), and the quantity \(\min\{|R^n|\}\) as \(R^n\), we fitted a power law \(D_n = D^* + a (R^n)^\alpha\). An excellent fit with eleven Farey levels \(n = 1, \ldots, 11\) starting with \(P/Q = \frac{5}{13}\) and \(P'/Q' = \frac{8}{21}\) was obtained. The number \(D^*\), which is our direct numerical estimate of the dimension of the set, was found to be \(0.868 \pm 0.002\), in agreement with Ref. 3. Surprisingly, the value of \(D_1\), an estimate based on only two gaps, was always very close to \(D^*\) (the deviation less than 1%). The result was invariant to the choice of \(P/Q\) and \(P'/Q'\) and can be applied to any interval of the staircase on Fig. 1. Moreover, the result is invariant to the choice of dynamical system \(\theta_{n+1} = f(\theta_n)\) as long

\begin{center}
\begin{figure}[h]
\includegraphics[width=0.5\textwidth]{fig1}
\caption{The mode-locking structure at \(K = 1\) for the map (1). The “devil’s staircase” is complete, and the complement of the mode-locked windings is of Lebesgue measure zero and universal fractal dimension \(D\) (Ref. 3).}
\end{figure}
\end{center}

\(\copyright 1985\) The American Physical Society

343
as \( f(\theta) \) has a cubic inflection point.

To understand the apparent universality we turned to a renormalization-group formulation. The analysis given below will provide convincing evidence that all the D's constructed in a small neighborhood of any "golden" winding number—one whose continued fraction ends in an infinite string of ones—will have the very same value of D. Since these numbers are dense in the interval \([0, 1]\), one has the first step in an argument that D is truly universal.

Previously, \(^{1b,c}\) the renormalization-group formulation has been used in this context to study the local scaling properties of golden winding numbers like \( w^* = (\sqrt{5} - 1)/2 \). \(^{10}\) A series of rational approximants \( w_n = F_n/F_{n+1} \) was constructed by using Fibonacci numbers \( F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, \quad F_1 = 1 \). Defining

\[
f_n(x) = f^{2n+1}(x) - F_n, \quad f^{(n)} = \alpha^n f_n(\alpha^{-n} x),
\]

one finds

\[
f^{(n+1)} = \alpha^n f^{(n)}(\alpha^{-2} x).
\]

In the limit \( n \to \infty \) one obtains the fixed-point equation

\[
f^*(x) = \alpha f^*(\alpha^{-2} x).
\]

The solution to this equation and its linearized version yields the relevant scaling parameters \( \alpha \) and \( \delta \), which are the exponents in x space and in parameter space, respectively. \(^{1b,c}\) Unfortunately in this formulation the dependence on the parameter \( \Omega \) is lost, and the universal mode-locking structure cannot be investigated. What is needed is a formulation that maintains the dependence on a parameter. Such a formulation is achieved by parametrization of the unstable manifold.\(^{11}\)

We construct the unstable manifold by starting with any given one-parameter family of functions \( f_\Omega(x) \) which have a cubic inflection point at \( x = 0 \). Since we found empirically that the set of interest was invariant to the choice of initial \( P/Q \) and \( P'/Q' \) of the Farey-tree construction, we can as well pick values according to two Fibonacci-number ratios. In this way we shall make full use of the work that has been done on the local scaling properties near the golden mean. Define now

\[
f^{(n)} = f^{(n+1)}(x) - F_n.
\]

Denoting by \( \Omega_n \) the value of \( \Omega \) for which \( f^{(n)}(x) \) has a superstable cycle with winding number \( F_{n-1}/F_n \), we define

\[
g^{(n)}(x) = c_n + f^{(n)}(x/c_n + 1).
\]

By construction \( x = 0 \) is a superstable fixed point of \( g^{(n)}(x) \). We want now the parameter range that spans the distance between the superstable cycle \( F_{n-1}/F_n \) and the next one \( F_n/F_{n+1} \) to be rescaled to the interval \([0, 1]\). We do so by turning \( g^{(n)}(x) \) into a one-parameter family by defining

\[
g^{(n)}_{\delta}(x) = c_n + f^{(n)}_{\delta}(x/c_n + 1),
\]

where \( \delta_n = \Delta_n/\Delta_{n-1} \) and \( \alpha_n = c_n + 1/c_n \). After infinitely many \( \alpha \) rescalings of \( x \) space around the inflection point \( x = 0 \), and infinitely many \( \delta \) shifts and rescalings in parameter space, we reach the universal one-parameter family of maps \( g_{\delta}(x) \) which lies on the unstable manifold and is invariant under rescaling and two-cycle composition. From Eq. (3) we get the exact result

\[
g_{\delta}(x) = \alpha g_1 + \beta \delta (\alpha g_1 + 1/\beta + \delta g_2(\alpha^2 x)).
\]

The normalization conditions are \( g_0(0) = 0, \quad g_1(0) = 1 \). We use now the universal object \( g_{\delta}(x) \) to investigate the structure of mode lockings. As noted before, for \( p = 0 \) \( g_{\delta}(x) \) has a superstable fixed point at \( x = 0 \). The range of \( p \) around zero for which \( g_{\delta}(x) \) still has a fixed point is the range of parameters for which the original map is locked on some ("infinitely" high) winding ratio \( w_n \). However, around \( p = 1 \) there is another locked state which corresponds to the next locked region in the sequence and the width of this region is scaled down by \( \delta \) compared to the first (we remember\(^{1a}\) that the meaning of \( \delta \) is that \( \Omega_n = \Omega GM + a/\delta^n \)). Around \( p = 1 + 1/\delta \) there is another, scaled down by \( \delta^2 \) compared to the first, etc. Thus, by studying the stability of the fixed point of \( g_{\delta} \) we can find an infinity of mode-locked states which are universally located. However, these are not all the locked ranges. In Fig. 2 we plot the largest locked ranges that can be obtained in the way just described, and also indicate some of those that do not fall into this category, since they correspond to winding numbers that are not \( F_n/F_{n+1} \). These are indeed needed to determine the fractal dimension \( D \). How can we get them from the univer-
sal object \( g_p(x) \)?

Consider for example the range denoted \((F_n + F_{n+2})/(F_{n+1} + F_{n+3})\). There the original map has an \((F_{n+1} + F_{n+3})\)-order cycle. The function

\[
q_p^{(n+3)}(x) = c f^{(n+2)}(\Delta_{n+1} x) f^{(n)}(\Delta_{n+1} x/c)
\]

will therefore have, for the range of \( p \) which falls in the desired region, a fixed point of order 1. We know from the Farey construction\(^8\) that this \( p \) falls somewhere between \( p = 0 \) and \( p = 1 + \delta \). The function \( q_p^{(n+3)} \) can also be written, using Eq. (1), as

\[
q_p^{(n+3)}(x) = (c/c_n + 3)q_p^{(n+2)}(c_n x)
\]

The range in \( p \) for which (6) has a fixed point is independent of \( c \); picking \( c = c_n + 1 \) we see that this range can be obtained from studying, in the limit \( n \rightarrow \infty \), the universal \( p \) range of stability of the fixed point of the function \( q_p^{[3]} \), which is obtained exactly from the composition

\[
q_p^{[3]}(x) = (1/\alpha^2)g_{q_p(x)}(\alpha^2 q_p(x)).
\]

Once this range of \( p \) is found, another will occur between \( p = 1 + \delta \) and \( p = 1 \), self-similarly placed but scaled down by \( \delta \), etc., etc.

Similarly the locking between \( F_{n+2}/F_{n+3} \) and \((F_n + F_{n+2})/(F_{n+1} + F_{n+3})\), which on the universal plot appears between \([2] \) and \([3] \), can be found by studying the stability of the fixed point of

\[
q_p^{[2,1]} = (1/\alpha^2)g_{q_p(x)}(\alpha^2 q_p^{[3]}(x)),
\]

whereas that between \( F_n/F_{n+1} \) \([0] \) and \((F_n + F_{n+2})/(F_{n+1} + F_{n+3}) \([1] \) from \( q_p^{[4]}(x) = q_p^{[3]}(g_p(x)) \).

Also these lockings will have infinite numbers of counterparts in the scaled-down regions on the right. All the universal locking intervals can be found by composing a pair of universal functions since every rational number can be expressed as a Farey composition of its two parent rationals.

In practice, an approximation to the function \( g_p(x) \) can be found relatively easily, starting from the definition (1) and any starting function \( f_0(x) \). We picked \( f_0(x) = x + (-1/2p) \sin(2\pi x) \) and the value \( \Omega_{n+1} = 0.606576201 \), which corresponds to winding number \( 2232/3200 \). Using the procedure described above we found the universal interval of \( p \) with the mode-locked structure shown in Fig. 2. The estimates of the fractal dimension were performed as before. With two gaps we find \( \tilde{\delta} = 0.6326 \), \( s_1 = 0.3425 \), \( s_2 = 0.2232 \), and \( D_1 = 0.858 \). Continuing this process we divided the universal set into more and more gaps and found \( D_1 \) for four Farey levels from \( \sum R^2 P = 1 \). Fitting as before \( D_1 \) by \( D = D^* + a (R^?) \) we obtained \( D^* = 0.867 \) in excellent agreement with the direct estimate. In this universal construction we know that \( \min_{\{R_i^\}} \) always comes from the gap closest to the golden mean and is asymptotically scaled down by \( \delta \) compared to \( \min_{\{R_i^\}} \). Accordingly this implies \( D_1 = D^* + a/\delta^2 \). Notice that this amounts to a geometric convergence of \( D_1 \). However, the convergence is rather slow \((\delta^2 > 1.1)\) and therefore a very precise statement about the rate of convergence cannot be made at this point.

A few comments are now in order. (i) More accurate results can probably be obtained by representation of \( g_p(x) \) as a double series expansion \( \sum_{i,j}\alpha_{ij} \times p^{i} x^{j} \), etc.\(^7\) For our purpose it was sufficient to work with \( g_p(x) \) obtained directly from Eq. (6) and use the previously determined\(^1\) values of the exponents \( \alpha \) and \( \delta \) as "input" for the calculation.

(ii) In analyzing experiments we suggest that the same procedure of estimating \( D \) would be taken. Two locked states with windings \( P/Q \) and \( P'/Q' \) should be identified, and the lockings which correspond to a Farey tree should be considered. \( \sum R^2 P = 1 \) and the fit can then be used to estimate the dimension. In fact, preliminary results in the context of convection experiments in an electrically conducting fluid appear to
agree with our predictions (private communication from A. Libchaber and J. Stavans).

(iii) The universality of the mode-locking structure discussed here can be continued to the complex plane, yielding a set analogous to the Mandelbrot set for the map \( z' = z^2 + c \). In Fig. 3 we show the set of complex parameters \( \Omega \) for which the complexified map \( f(z) = z + \Omega - (1/2\pi) \sin(2\pi z) \) has locked solutions. We convinced ourselves numerically that this set has similar universal properties to the ones discussed above. In particular, the regions obtained by stretching up the set contained between winding numbers \( F_n/F_{n+1} \) and \( F_{n+1}/F_{n+2} \) seem to result in an invariant set. One can also see that the shape of any “egg” has a degree of universality. In particular the ratio of the width on the real axis to the height as measured by the point linking the “egg” to its largest leaf is apparently constant \((\sim 1.1)\).

This work has been supported in part by the U. S. Office of Naval Research and by the Materials Research Laboratory, University of Chicago. One of us (I.P.) acknowledges partial support by the Minerva foundation, Munich, Germany and by a National Science Foundation grant. One of us (M.H.J.) acknowledges travel support from NATO Science Fellowship Programme. One of us (P.C.) acknowledges support from the U. S. Department of Energy under Contract No. DE-AC02-83-ER13044. We benefitted from discussions with Mitchell Feigenbaum, Albert Libchaber, Boris Shraiman, and Joel Stavans.

(a) Permanent address: Theoretical Physics, Chalmers, Göteborg, Sweden.
(b) Permanent address: Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel.
(c) Scott J. Shenker, Physica (Amsterdam) 5D, 405 (1982).
(f) A. P. Fein, M. S. Heutmaker, and J. P. Gollub, Phys. Scr. T9, 79 (1985). Recently, however, preliminary experimental results by A. Libchaber and J. Stavans (private communication) seem to support the theoretical predictions in Ref. 1.
(l) Shraiman (unpublished) introduced the Farey tree in this context.
(m) P. Cvitanović and J. Myrheim, to be published.
(p) A different, ergodic renormalization-group formulation was performed by J. D. Farmer and I. Satija, Phys. Rev. A 31, 3520 (1985).
(r) Each mode locking is conveniently labeled by the continued-fraction \([k_1, k_2, k_3, \ldots] \) representation of the winding number in the corresponding Farey tree. \( F_n/F_{n+1} \) is the root of the tree \([0]; F_{n+1}/F_{n+2} \) is the next level, \([1]; (F_{n+2}+F_n)/(F_{n+1}+F_{n+2}) \) corresponds to \([2]; F_{n+3}/F_{n+4} \) to \([1, 2], \) and so on.
(s) M. J. Feigenbaum and G. Gunaratne, unpublished.