1. Scaling, Universality and Operator Algebras

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Introductory Note ........................................ 2
I. Critical Points and Thermodynamic Singularities ....... 2
   A. The Ising model ........................................ 3
   B. Other problems ......................................... 6
   C. Critical singularities ................................... 8
   D. The importance of correlation functions .......... 10
   E. Droplet picture of correlation behaviour .......... 10
II. Mean Field Theory ........................................ 12
    A. Results ................................................ 12
    B. Generalizations ....................................... 15
    C. Failure of mean field theory ....................... 15
III. The Roots of the Theory: Universality ................. 16
     A. Fields and operators ................................. 16
     B. Statement of universality hypothesis ............ 17
     C. A simple example ..................................... 18
     D. Additional universality statements ............... 19
IV. The Roots of the Theory: Scaling ......................... 20
     A. The irrelevance of the length scale ............... 20
     B. A formal description of the length transformation 20
     C. Fixed points .......................................... 23
     D. Deviations from the fixed point .................. 24
     E. Scaling results for the free energy .............. 25
     F. A simple example of the theory .................... 26
V. Scale Transformations for Correlation Functions ....... 28
   A. General discussion ..................................... 28
   B. Formal theory ......................................... 28
   C. Connections with universality ....................... 31
   D. Consequences .......................................... 31
References ................................................... 33
Introductory Note

There are three identifiable periods in the development of the theory of critical point behaviour. The first and longest period began with Van der Waal's development of a mean field theory approach to the liquid–gas phase transition. This approach proved to be tremendously rich and extendable to the description of a whole series of different phase transitions. A beautiful extension and culmination of this approach is found in the BCS theory of superconductivity. However, in the decade preceding 1965 both numerical calculations and experiments showed with increasing clarity that the mean field idea was quantitatively incorrect in the neighbourhood of thermodynamic critical point.

The next era in the development of the field started around 1965 with the incorporation of these facts into more or less phenomenological theories of critical behaviour. These phenomenological approaches can all be understood in terms of the concept of scaling invariance, which was brought forward at roughly the same time. The scaling concept is one of the ideas described in detail in this paper.

In this same period, another key idea in critical phenomena theory, universality, was extensively developed. The idea of universality is that apparently dissimilar systems show considerable similarities near their critical points. It is an old idea dating back to the nineteenth century, but its mathematical formulation has led to a considerably better understanding of the underlying nature of critical phenomena and also to a much more useful analysis of experiments. The recent developments in the concepts of universality form another main section of this paper.

The most recent period in the theory was opened with a brilliant series of papers by K. Wilson and co-workers. Wilson essentially took the semi-phenomenological concepts of scaling and universality and converted these ideas into real calculations of critical point behaviour. Literally hundreds of other calculations have followed upon this breakthrough. In addition, F. Wegner has shown how the concepts of universality and scaling naturally follow from Wilson's calculational method.

In this paper, I shall utilize the hindsight offered by Wegner's formulation of Wilson's approach to explain and develop the concepts included in the theory of critical phenomena.

I. Critical Points and Thermodynamic Singularities

In this section of the paper, the basic concepts of critical phenomena theory are introduced, using the notation and example of the Ising model.
A. The Ising model

1. Definition

Given a lattice of points \( \vec{r} \) at each point there is a spin \( \sigma(\vec{r}) \) which can take on two values:

\[
\sigma(\vec{r}) = \pm 1.
\]

There is an attractive interaction between spins at neighbouring sites which gives an energy \(-J\) if the spins point in the same direction and \(+J\) if they point in opposite directions. Also, there is an external magnetic field \(H\) which gives a negative energy \(-H\) if the spin is lined up with the field and \(+H\) if it is not. In total, then, the energy of the system is:

\[
\mathcal{H} = -J \sum_{\text{nearest neighbours}} \sigma(r)\sigma(r') - H \sum_r \sigma(r)
\]

All thermodynamic properties are defined by the free-energy function, \(\mathcal{F}(\sigma)\) defined by

\[
\exp[+ \mathcal{F}(\sigma)] = \exp[-\mathcal{H}/kT]
\]

where \(k\) is the Boltzmann constant and \(T\) is the absolute temperature. Here, the free energy function \(\mathcal{F}(\sigma)\) depends upon two extensive operators

\[
S_1 = \sum_r \sigma(r)
\]

\[
S_2 = \sum_{\text{nearest neighbours}} \sigma(r)\sigma(r')
\]

as

\[
\mathcal{F}(\sigma) = hS_1 + KS_2
\]

The quantities \(S_1\) and \(S_2\) are termed extensive operators because they are sums over the entire lattice. In contrast \(\sigma(r)\) and

\[
\delta(r_1) = \sum_{r_2, \text{nearest neighbours}} \sigma(r_1)\sigma(r_2)
\]

are called local operators. These local operators are, of course, the summands in the definitions (1.4) and (1.5) of the extensive operators.

The two parameters \(h\) and \(K\) appearing in the definition (1.6) of \(\mathcal{F}(\sigma)\) are given by

\[
h = H/kT
\]

\[
K = J/kT.
\]

In general, we shall utilize the notation \(S_a\) for the \(a^{th}\) extensive operator.
in the theory. The corresponding local operators will be written as \( s_a(r) \),
while the parameters in the free energy will be denoted by \( K_a \). In particular,
\( K_1 = h \) and \( K_2 = K \).

In terms of these quantities, we can define a free energy density \( f(K) \), as
\[
Z(K) = \exp \left[ N f(K) \right] = \sum_{\sigma} e^{\mathcal{F}(\sigma)}
\]
(1.10)

Here \( N \) is the number of spin sites and \( K \) is a vector with components
\((K_1, K_2) = (h, k)\).

2. Thermodynamic averages

The average of any function of the spins \( \Theta(\sigma) \) is given by
\[
\langle \Theta(\sigma) \rangle = \frac{1}{Z} \sum_{\sigma} \Theta(\sigma) \exp(\mathcal{F}(\sigma))
\]
(1.11)

The most important averages are the expectation value of the local operators,
\[
m_a(K) = \langle s_a(r) \rangle = \frac{\partial f(K)}{\partial K_a},
\]
(1.12)

the correlation functions,
\[
g_{ab}(r, K) = \langle s_a(0)s_b(r) \rangle
\]
(1.13)

and the generalized susceptibility
\[
\chi_{ab}(K) = \langle s_a(r)s_b(r) \rangle - \langle s_a(r) \rangle \langle s_b(r) \rangle.
\]
(1.14)

The expression (1.12) which relates the average of the local densities with
derivations of the free energy density follows directly from eqn (1.10) and the
definition (1.11) of an average since
\[
S_a = \sum_r s_a(r).
\]
(1.15)

In an analogous fashion, the susceptibility (1.14) can also be related to
derivatives of the free energy density as
\[
\chi_{ab}(K) = \frac{\partial^2 f(K)}{\partial K_a \partial K_b}
\]
(1.16)

Moreover, \( \chi \) can be written in terms of the connected part of the correlation
function
\[
g_{ab}^c(r, K) = g_{ab}(r, K) - \langle s_a(0) \rangle \langle s_b(r) \rangle
\]
(1.17)

\[
\chi_{ab}(K) = \sum_r g_{ab}^c(r, K).
\]
(1.18)
1. Scaling, universality and operator algebras

The most important averages and correlations are those involving the spin-variable. We use a special notation for these, i.e.,

\[ \langle \sigma(r) \rangle = m(K) \quad (1.19) \]

\[ \langle \sigma(0)\sigma(r) \rangle = g(r, K) \quad (1.20) \]

\[ \langle \sigma(0)\sigma(r) \rangle - [m(K)]^2 = g'(r, K). \quad (1.21) \]

Finally, the heat capacity, \( C_n \), is defined as

\[ C_n = \frac{\partial^2 \langle s_s(r) \rangle}{\partial K^2} = \frac{\partial^2}{\partial K^2} f(K). \quad (1.22) \]

3. Qualitative description of behaviour

For large \( T(K \to 0) \) and \( h = 0, \langle \sigma \rangle = 0 \). For small \( T \), there is a tendency for spins to line up. If the dimensionality of the lattice is greater than 1, then for \( T < T_c(K > K_c) \),

\[ \langle \sigma \rangle = \pm m(0+, K) \quad (1.23) \]

since the spins can line up in either the positive or negative direction.

The system can jump from the state with

\[ \langle \sigma \rangle = + m(0+, K) \quad (1.24a) \]

to the state with

\[ \langle \sigma \rangle = - m(0+, K). \quad (1.24b) \]

This jump is called a first order phase transition. Thus at \( h = 0 \), the state of the system is not uniquely defined as a function of \( K \).

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Fig. 1.1. Magnetization vs temperature (schematic) for various values of \( h \). Below the \( h = 0 \) curve, there are no stable thermodynamic states.

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Hence, the sta the ter energy
If $h \neq 0$, the state is uniquely defined. For $h > 0$, $\langle \sigma \rangle > 0$; for $h < 0$, $\langle \sigma \rangle < 0$. These results are shown in Fig. 1.1.

If we look at behaviour as a function of $T$ and $h$, we see, in Fig. 1.2, that the first order phase transition produces a line of singularities which culminate in the critical point.

![Fig. 1.2. The phase diagram of the Ising model.](image)

The critical point is the point at which the first order phase transition disappears. Since

$$\sigma(h = 0+, K) = 0 \text{ for } K < K_c$$

and

$$\sigma(h = 0+, K) \neq 0 \text{ for } K > K_c$$

the critical point is necessarily a point of non-analyticity for the system.

In this paper we are interested in the behaviour in the neighbourhood of this critical point.

B. Other problems

The most natural parameters to use in the description of the critical behaviour of the Ising model are the magnetic field variable $h$ and

$$t = (T - T_c)/T_c.$$

In the free energy function $\mathcal{F} \{\sigma\}$, $h$ is paired with the magnetization density $\sigma(r)$ and $t$ (or $K$) with the energy density $\mathcal{E}(r)$. As Griffiths and Wheeler have pointed out, these parameters play a very natural role in a phase diagram like Fig. 1.2. Changes in the parameter $h$ drive the system across the phase boundary. Changes in $t$ produce motion parallel to the phase boundary. Hence, it is these two variables which produce the significant changes in the state of the system in the thermodynamic region. Griffiths introduced the term, fields, to describe parameters like these which appear in the free energy and which vary continuously across a first order phase transition.
<table>
<thead>
<tr>
<th>Phase transition problem</th>
<th>First field = $h$</th>
<th>First operator order parameter</th>
<th>Second field = $t$</th>
<th>Second operator</th>
<th>Third field</th>
<th>Third operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ising model dimensionality $\geq 2$</td>
<td>$h$</td>
<td>$\sigma(r)$</td>
<td>$(T - T_c)/T_c$</td>
<td>Energy density</td>
<td>Next nearest neighbour coupling</td>
<td>Next neighbour energy density</td>
</tr>
<tr>
<td>Ising model dimensionality = 1 Ferromagnetic transition</td>
<td>$h$</td>
<td>$\sigma(r)$</td>
<td>$e^{-k}$</td>
<td>Energy density</td>
<td>Staggered field</td>
<td>Staggered magnetization density</td>
</tr>
<tr>
<td>Antiferromagnetic transition</td>
<td>Staggered magnetic field $^{(b)}$</td>
<td>Magnetization density $^{(a)}$</td>
<td>$(T - T_c)/T_c$</td>
<td>Energy density</td>
<td>Magnetic field</td>
<td>Magnetization density</td>
</tr>
<tr>
<td>Liquid–gas</td>
<td>Chemical potential</td>
<td>Particle density minus initial density</td>
<td>$(T - T_c)/T_c$</td>
<td>Energy density $^{(c)}$</td>
<td>Parameter measuring size of quantum effects</td>
<td>—</td>
</tr>
<tr>
<td>Superconducting Superfluid</td>
<td>Not physical</td>
<td>$\Delta(r)$$^{(d)}$</td>
<td>$(T - T_c)/T_c$</td>
<td>Energy density</td>
<td>Pressure</td>
<td>Density</td>
</tr>
<tr>
<td></td>
<td>Not physical</td>
<td>$\psi(r)$$^{(d)}$</td>
<td>$(T - T_c)/T_c$</td>
<td>Energy density</td>
<td>Pressure</td>
<td>Density</td>
</tr>
</tbody>
</table>

(a) These quantities are scalars when there is an easy axis of magnetization. When there is an easy plane, they are two dimensional vectors; when (as in the Heisenberg model) the zero field magnetization may point in any direction in space, then they are vectors.

(b) The antiferromagnetic lattice splits into two sublattices. Staggered quantities point in opposite directions on the sublattices. The quantities involved may be scalars or vectors as in note (a).

(c) A linear combination of this field or operator must be taken with the first one to provide changes orthogonal to the phase boundary.

(d) An annihilation operator for Cooper pairs.

(e) The wave field annihilation operator.
Other phase transition problems can equally well be defined in terms of these two fields:
(a) A field $h$, which drives the system across the phase boundary and vanishes at criticality;
(b) A field $t$, which moves the system along the phase boundary and also vanishes at the critical point. Table 1.1 lists the fields and conjugate operator densities which we shall use in the description of different phase transition problems. For later reference, we have included another set of fields and operators in the last two columns of Table 1.1. The reader is asked to pass these over for the time being.

C. Critical singularities
The theory must explain a very rich collection of singularities. These singularities are generally of the form of power laws. For example, at $h = 0$, in the variable $t$ we see singularities of the form $t^{-\lambda}$ or logarithms, which are limits of these power law singularities, i.e.,
\[
\ln t^{-1} = \lim_{\lambda \to 0} \frac{t^{-\lambda} - 1}{\lambda}.
\]
Exponents like $\lambda$, which appear in critical singularities are termed critical indices. Table 1.2 gives a definition of the critical indices used in this paper.

<table>
<thead>
<tr>
<th>Critical index</th>
<th>Definition</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$m \approx \pm (-t)^{\beta}$</td>
<td>$t &lt; 0$, $h = 0$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$x \approx t^{-\gamma}$</td>
<td>$t &gt; 0$, $h = 0$</td>
</tr>
<tr>
<td>$\gamma'$</td>
<td>$x' \approx (-t)^{-\gamma'}$</td>
<td>$t &lt; 0$, $h = 0$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$m \approx h^{1/\delta}$</td>
<td>$t = 0$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$C_{\alpha} \approx t^{-\alpha}$</td>
<td>$t &gt; 0$, $h = 0$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\xi \approx t^{-\nu}$</td>
<td>$r \to \infty$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>$g(r) \approx (1/r^{2x_0})$</td>
<td>$t = h = 0$, $r \to \infty$</td>
</tr>
<tr>
<td>$x_s$</td>
<td>$g_2(r) \approx (1/r^{2x_s})$</td>
<td>$t = h = 0$, $r \to \infty$</td>
</tr>
</tbody>
</table>

Table 1.3 show some of the values of the indices observed in experiments and numerical calculations. Here $d$ represents the spatial dimensionality of the lattice. The last column (MFT) refers to mean field theory. We shall describe this approximation in the next section.

The important points to notice from Table 1.3 are the following:
1. All indices seem to vary smoothly with dimensionality;
Table 1.3. Values of critical indices.

<table>
<thead>
<tr>
<th>Critical index</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
<th>Fluids $d = 3$</th>
<th>Ferromagnetic models $d = 3$</th>
<th>Ising model $d = 4^{(a)}$</th>
<th>Ising model $d &gt; 4$</th>
<th>MFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{3}{4}$</td>
<td>$-\frac{9}{4}$</td>
<td>$-\frac{31}{4}$</td>
<td>$0.34$</td>
<td>$0.33$</td>
<td>$-\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$2$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{5}{4}$</td>
<td>$1.250$</td>
<td>$1.22 \pm 0.05$</td>
<td>$1.32$</td>
<td>$1.38$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\infty$</td>
<td>$15$</td>
<td>$5.0\pm 0.2$</td>
<td>$4.4 \pm 0.2$</td>
<td>$5$</td>
<td>$5$</td>
<td>$3$</td>
<td></td>
</tr>
<tr>
<td>$\xi$</td>
<td>$0^{(a)}$</td>
<td>$0.12$</td>
<td>$0.1$</td>
<td>$0.640$</td>
<td>$0.65$</td>
<td>$0.675$</td>
<td>$0.7$</td>
<td>$0^{(a)}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$2$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$\pm 0.003$</td>
<td>$\pm 0.05$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0.55$</td>
<td>$0.5$</td>
<td>$0.51$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>$-\frac{1}{2}$</td>
<td>$\approx \frac{1}{2}$</td>
<td>$\approx \frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

(a) Extra logarithms can appear at $d = 4$.
(b) A discontinuity at $t = 0$.
(c) A logarithmic singularity.
2. The real fluids show indices close to but not exactly equal to the indices of the three-dimensional Ising model;
3. The ferromagnetic models show a dependence of the critical indices upon whether there is any easy axis of magnetization, or an easy plane, or the magnetization can point in any direction;
4. There is a very rich set of data to be explained by the theory.

D. The importance of correlation functions
One of the most dramatic features of critical behaviour is the infinity in the magnetic susceptibility at criticality. Notice that this susceptibility is the sum over sites of a correlation function which is strictly bounded to be < 1, since
\[ \chi = \sum_r g'(r, h, t) = \sum_r [\langle \sigma_0 \sigma_r \rangle - \langle \sigma \rangle^2]. \]
The infinity at criticality is achieved because at criticality, \( g(r) \) is not integrable. Instead, it is a power of the separation distance
\[ g'(r, 0, 0) \sim \frac{1}{r^{\Delta}} \]
for large \( r \). We conclude that the characteristic infinities of thermodynamic functions at the critical point are reflections of correlations which extend over infinite distances in space.

At any finite distance from criticality, correlations tend to fall off with exponential rapidity as \( r \to \infty \), i.e.
\[ \lim_{r \to \infty} g'(r, h, t) \sim e^{-r/\xi} \times \text{weaker function of } r. \] (1.25)
Here \( \xi \) has the physical significance of being a range of correlations within the system. This correlation length tends to diverge at the critical point. In particular, at \( h = 0 \),
\[ \xi \sim t^{-\nu}. \] (1.26)

The divergence, (1.26), in the correlation length is the fundamental source of all thermodynamic singularities at the critical point.

E. Droplet picture of correlation behaviour
The physical source of this correlation length divergence may be seen by considering fluctuations for \( t < 0 \) and \( h = 0 \). Begin at low temperatures. Assume that the system has a net magnetization pointing in the +z direction. A fluctuation may drive a region of the material into a state in which the magnetization points in the "wrong" direction. This droplet of the "wrong" phase has the same energy per unit volume as the material with the "right"
magnetization. However, there is an extra free energy added to the system proportional to the area of the region and to a cost per unit area for forming the boundary

\[
\text{free energy cost} \sim \text{area} \times \frac{\text{energy}}{\text{unit area}}
\]  
(1.27)

The formation of such a drop will be very unlikely if the cost in free energy is much greater than \( kT \). For this reason, we find

\[
\text{area} \leq \frac{kT}{\text{energy}} \text{ (unit area)}
\]  
(1.28)

However, as criticality is approached, the difference in magnetization between the two different phases gets smaller and smaller. Hence the energetic cost per unit area of producing a region of the wrong phase approaches zero. For this reason, the area of a droplet and its radius both can get very, very large. Critical phenomena are connected with large-scale but weak fluctuations in the magnetization.

So far, our picture of critical fluctuations is like that in Fig. 1.3. Droplets

![Fig. 1.3. Droplet picture of the critical region.](image)

with spin down of all sizes up to a maximum size \( \xi \) appear near the critical point. The size, \( \xi \), is determined by condition (1.28).

However, this picture is incomplete. Each fluctuating region is also a nearly-critical system.

Fluctuations appear within the droplets. And within these fluctuations
yet more appear. Hence, each droplet in Fig. 1.3 has a characteristic appearance like that shown in Fig. 1.4. This clustering of droplets within droplets appears until a purely microscopic scale of distances is reached.

From this picture we conclude that critical phenomena are connected with fluctuations over all length scales between \( \xi \) and the microscopic distance between particles.

II. Mean Field Theory

A. Results

A first qualitative picture of near-critical behaviour can be obtained by neglecting fluctuation phenomena. In one sense, fluctuations are at the heart of critical phenomena. However, in another sense, one can discuss phase transitions by considering the important effect to be the lining up of spins to form an order in the system. Then, one can take into account the average order through \( \langle \sigma \rangle \). Consider a case in which the magnetic field, \( h_r \), varies with \( r \). Then,

\[
\exp \left[ \frac{\mathcal{F} \langle \sigma \rangle}{\xi} \right] = \exp \left[ K \sum_{\langle r' \rangle} \sigma(r)\sigma(r') + \sum_r h_r \sigma(r) \right].
\]  

(2.1)

Now focus your attention upon one spin, the one at \( r \). Replace every other spin by its average. Then, the free energy function (2.1) becomes

\[
\exp \left[ \mathcal{F} \{ \sigma \} \right] \rightarrow \text{const} \times \exp \left[ \sigma(r) h_r^{\text{eff}} \right]
\]

(2.2)
where the effective or mean field is
\[
h^\text{eff}_r = h_r + \sum_{r' \text{ neighbours of } r} K \langle \sigma(r') \rangle \tag{2.3}
\]
From eqn (2.2) the average of \(\sigma(r)\) is
\[
\langle \sigma(r) \rangle = \tanh h^\text{eff}_r. \tag{2.4}
\]
If \(\sigma(r)\) varies slowly in space, eqn (2.3) may be replaced by
\[
h^\text{eff}_r = h_r + zK \langle \sigma(r) \rangle + cV^2 \langle \sigma(r) \rangle \tag{2.5}
\]
where \(z\) is the number of nearest neighbours and
\[
c = K \sum_{r'} \frac{(r' - r)^2}{d} \tag{2.6}
\]
with \(d\) being the dimensionality. The sum in 2.6 extends over all nearest neighbours to \(r\). If eqn (2.4) is expanded to first order in \(h_r\) and \(c\) and to third in \(K\langle \sigma \rangle\) we find
\[
h_r = (1 - zK)\langle \sigma(r) \rangle - \frac{1}{3}\langle \sigma(r) \rangle^3(zK)^3 - cV^2 \langle \sigma(r) \rangle. \tag{2.7}
\]
The condition for criticality in eqn (2.7) is that \(h \to 0\) and
\[
t = 1 - zK
\]
be zero. Hence we can replace (2.7) by
\[
[t + \frac{1}{3}\langle \sigma(r) \rangle^2 - cV^2]\langle \sigma(r) \rangle = h_r. \tag{2.8}
\]
Equation (2.8) is our basic mean field theory approximation.
If \(h\) is independent of \(r\), we derive the equation of state
\[
[t + \frac{1}{3}\langle \sigma \rangle^2]\langle \sigma \rangle = h. \tag{2.9}
\]
From this result, we can derive all the thermodynamic properties of this mean field approximation. For example, if \(h = 0\) and \(t > 0\), we find \(\langle \sigma \rangle = 0\). But if \(t < 0\), i.e. we are below the critical temperature, there exists the solution
\[
\langle \sigma \rangle = \pm (-3t)^{1/2} \tag{2.10}
\]
so that \(\beta = \frac{1}{t}\). The indices \(\gamma\), \(\delta\), and \(\gamma'\) are derived in similar fashion. Since
\[
\langle \sigma \rangle = (\partial f / \partial h)_t,
\]
we can derive an expression for \(f\) by writing
\[
f'(\langle \sigma \rangle, t) = f(\sigma, t) - \langle \sigma \rangle h \tag{2.11}
\]
so that
\[
\frac{\partial f'}{\partial \langle \sigma \rangle} \bigg|_t = -h = -[t\langle \sigma \rangle + \frac{1}{3}\langle \sigma \rangle^3]. \tag{2.12}
\]
The last equality follows from eqn (2.9). Equation (2.12) may be integrated to yield
\[
f' = -\left[ \frac{t\langle \sigma \rangle^2}{2} + \frac{1}{12}\langle \sigma \rangle^4 \right].
\] (2.13)

Then, the specific heat is
\[
C_h = \left. \frac{\partial^2}{\partial t^2} f(h, t) \right|_{h}
\]
\[
= \left. \frac{\partial^2}{\partial t^2} \left[ f'\langle \sigma \rangle, t \right] + \langle \sigma \rangle h \right] \right|_{h}
\]
\[
= \left. \frac{\partial}{\partial t} \left( \frac{\partial f'}{\partial t} \langle \sigma \rangle + \frac{\partial}{\partial t} \frac{\partial^2}{\partial t^2} \langle \sigma \rangle \right) \right|_{h}
\]
\[
= \left. h \frac{\partial^2}{\partial t^2} \langle \sigma \rangle \right|_{h} - \frac{1}{2} \left. \frac{\partial}{\partial t} \frac{\partial}{\partial t} \langle \sigma \rangle^2 \right|_{h}
\]

at \( h = 0 \), this result reduces to
\[
C_h = 0 \quad \text{for } t > 0
\]
\[
= +\frac{2}{3} \quad \text{for } t < 0.
\] (2.14)

Thus the specific heat shows a discontinuity at \( t = 0 \) in this mean field approximation.

Finally, the spin–spin correlation function obeys
\[
\langle \sigma(0)\sigma(r) \rangle = \langle \sigma \rangle^2 + \frac{\delta \langle \sigma(r) \rangle}{\delta h(0)}
\]
\[
= \langle \sigma \rangle^2 + g'(r, h, t)
\] (2.15)

From eqn (2.8), the connected part of the correlation function obeys
\[
[t + \frac{1}{2}\langle \sigma(r) \rangle^2 - cV^2]g'(r, h, t) = \delta_{r0}.
\]

At \( h = 0 \), at \( t > 0 \) this gives a Fourier transform
\[
(t + cq^2)\hat{g}(q, h, t) = 1.
\]

Then
\[
\hat{g}(r, h, t) \bigg|_{h=0} = \int \frac{d^dq}{(2\pi)^d} \exp i\mathbf{q} \cdot \mathbf{r} \frac{1}{t + cq^2}
\]

In three dimensions we then find
1. Scaling, universality and operator algebras

\[ \hat{g}(r, h, t) \bigg|_{h=0} = \frac{1}{4\pi rc} e^{-r/\xi}; \quad \xi = \sqrt{c/t}. \]  

(2.16)

Hence \( x_o = \frac{1}{2} \) and \( v = \frac{1}{2} \). More generally, to an order of magnitude

\[ \hat{g}(r, h, t) \sim \frac{1}{c^{d-2}} e^{-r/\xi}. \]  

(2.17)

so that \( x_o = \frac{1}{2}(d - 2) \) in \( d \)-dimensions.

B. Generalizations

For most phase transitions, one can define an order parameter analogous to \( \langle \sigma(r) \rangle \). This order parameter has the following properties:

1. It jumps discontinuously across the first order phase transition;
2. As criticality is approached, this jump gets smaller and smaller;
3. Large values of the order parameter imply that one is far from criticality.

One can also usually define a free energy density \( f(h_o, t) \) such that

\[ \frac{\delta}{\delta h_o} \sum_{r'} f(h_o, t) = \langle \sigma(r) \rangle \]  

(2.18)

where \( \langle \sigma(r) \rangle \) is the order parameter. Then it is also possible to define

\[ f'(\langle \sigma \rangle, t) = f(h_o, t) - \langle \sigma(r) \rangle h_o \]  

(2.19)

so that

\[ \sum_{r'} \frac{\delta}{\delta \langle \sigma(r) \rangle} f'(\langle \sigma(r) \rangle, t) = -h_o. \]  

(2.20)

The mean field theory is the assumption that one can expand the free energy function \( f' \) in a power series in \( \langle \sigma(r) \rangle \) as

\[ f'(\langle \sigma(r) \rangle, t) = -\frac{t}{2} \langle \sigma(r) \rangle^2 + b\langle \sigma(r) \rangle^4 + c[\nabla \langle \sigma(r) \rangle]^2 \]  

(2.21)

All the consequences of mean field theory follow directly from this expansion assumption.

C. Failure of mean field theory

In one sense, mean field theory is a great success. It does give first order phase transitions and also a qualitative indication of the types of singularities which might be expected in the second order phase transition.

In a quantitative sense, however, it is a complete failure in the critical region. All the critical indices derived from mean field theory are wrong in two and three dimensions. The source of this failure is clear. Critical phe-
nomina involve fluctuations. Mean field theory is based upon the assumption of small fluctuations.

To see this quantitative failure, we shall use mean field theory to predict its own shortcomings. For $t < 0$ and $h = 0$, it is exactly true that

$$C_h \sim -\frac{\partial}{\partial t} \langle \sigma^2 \rangle$$

(2.22)

where $a$ is the distance between two nearest neighbours. Our free-energy calculation approximates the expression (2.22) by

$$C_h = -\frac{\partial}{\partial t} \langle \sigma^2 \rangle^2 = + 3.$$

However, a more careful calculation based upon mean field theory would give

$$C_h = -\frac{\partial}{\partial t} \left[ \langle \sigma^2 \rangle + g(r = a, h = 0, t) \right]$$

$$= \frac{3}{2} - \frac{1}{2} \frac{\partial}{\partial t} \int \frac{d^d q}{(2\pi)^d} \frac{\exp iq \cdot a}{t + q^2 c}$$

$$= \frac{3}{2} + \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\exp iq \cdot a}{(q^2 c + t)^{d/2}}$$

(2.23)

Therefore

$$C_h - \frac{3}{2} \sim \frac{q^d}{(q^2 c)^{d/2}} \sim (q^2 c)^{d/2}$$

$$\sim \frac{1}{c^{d/2}} (t)^{-4 - d/2}$$

If the mean field theory is to be correct, the right-hand side must be small. But this term diverges as $t \to 0$ whenever

$$d < 4.$$

(2.24)

Hence we know that mean field theory must fail for all dimensionalities less than 4.

III. The Roots of The Theory: Universality

A. Fields and operators

Since critical phenomena arise from long-ranged correlations, it is reasonable to expect that some of the details of the interatomic potential might be quite irrelevant to the behaviour in the critical region. Thus, for example, it is
usually asserted that the values of the critical indices are independent of lattice structure. Similarly we expect that if the Hamiltonian is a mixture of nearest-neighbour and next nearest neighbour interactions the critical behaviour is independent of the exact mixing ratio.

To convert these qualitative statements into quantitative form, we write the basic free energy function for the system as

\[ \exp[\mathcal{F}(\sigma)] = \exp[\mathcal{F}^*(\sigma)] + \sum \lambda_i S_i(\sigma) \]  

(3.1)

Here \( \mathcal{F}^*(\sigma) \) is a free energy function which produces some kind of critical point behaviour. The fields \( \lambda_i \) represent some kinds of deviation from this critical point. For example, \( \lambda_1 \) might be the magnetic field; \( \lambda_2 \) might be \( T \) — the temperature deviation from criticality — and so forth. The \( S_i \)'s are then operators (i.e., functions of the \( \sigma \)'s) which are thermodynamically conjugate to the \( \lambda_i \)'s. Thus \( S_1 \) might be the total magnetization, \( S_2 \) the energy, etc. Each \( S_i \) is translationally invariant and may be written as a sum over all lattice sites of a local density \( s_i(\vec{r}) \), i.e.,

\[ S_i = \sum_{\vec{r}} s_i(\vec{r}). \]  

(3.2)

Thus for example, \( s_1(\vec{r}) \) might be \( \sigma(\vec{r}) \), \( s_2(\vec{r}) \) might be

\[ \sum_{\vec{a}} \sigma(\vec{r} + \vec{a}) \]

where the sum over \( \vec{a} \) is a sum over vectors to nearest neighbours, etc.

The most important descriptors of critical behaviour are the free energy per site, \( f(\vec{h}) \), the average of the operators, and their correlations. If \( N \) is the number of sites, we define

\[ f(\vec{h}) = \frac{1}{N} \ln \sum_{\{\sigma\}} \exp[\mathcal{F}(\sigma)] \]

\[ m\hspace{1pt}_a(h) = \langle s_a(r) \rangle \]  

(3.3)

\[ g_{0\hspace{2pt}b}(\vec{r}, \vec{h}) = \langle s_0(0) s_b(\vec{r}) \rangle \]

and notice that we can define a generalized susceptibility \( \chi_{ab}(\vec{h}) \) in three different ways:

\[ \chi_{ab} = \frac{\partial^2 f}{\partial \lambda_a \partial \lambda_b} = \frac{\partial m_a}{\partial \lambda_b} = \sum_{\vec{r}} \left[ g_{ab}(\vec{r}, \vec{h}) - \langle s_a \rangle \langle s_b \rangle \right]. \]  

(3.4)

B. Statement of universality hypothesis

In its simplest terms, the universality hypothesis is the statement that all critical problems may be divided into classes differentiated by:

(a) The dimensionality of the system;
(b) The symmetry group of the order parameter; and
(c) Perhaps other criteria.
Within each class, the critical properties are supposed to be identical or, at worst, to be a continuous function of a very few parameters.

To convert this physical statement into mathematical form, we consider a reference problem characterized by fields \( h_n \), local operators \( s_n(r) \), a free-energy per site \( f(h) \), an expectation value of the fields \( m_\lambda(h) \) and a set of correlation functions \( g^{(2)}_{\lambda \beta}(r, h) \). A comparison problem is defined by fields \( h_n^\prime \), operators \( s_n^\prime(r) \), free-energy per site \( f(h') \), an expectation value \( m_\lambda(h') \) and correlation functions \( g^{(2)}_{\lambda \beta}(r, h') \). Then, the two problems are said to be in the same universality class if there exists a transformation upon problem \( I' \) which will convert its solution into a form identical with the solution of problem \( I \). In particular, the transformation is hypothesized to take the form

\[
\begin{align*}
    h_n' &\rightarrow h_n' = b_{\lambda \beta} h_n^\prime \\
    s_n^\prime(r) &\rightarrow s_n^\prime(r) = s_n^\prime(r)c_{\lambda \beta} \\
    r &\rightarrow r' = r/l
\end{align*}
\]

with \( b_{\lambda \beta} \), \( c_{\lambda \beta} \), and \( l \) being analytic functions of \( h_n \) for small values of \( h_n \). Here and below we use a summation convention for repeated Greek indices.

The exact meaning of the identity between the two problems is that the free-energies per site are related by

\[
f(h') = b_0 f(h) + \text{a non-singular function of } h'
\]

where \( b_0 \) is analytic in \( h \), and that the operator averages are given by

\[
\langle s_n'(r) \rangle \equiv m_{\lambda \beta}(h')c_{\lambda \beta} = m_{\lambda}(h) + \text{a non-singular function of } h_n.'s
\]

Moreover, the correlation functions are related by

\[
g^{(2)}_{\lambda \beta}(r, h') = g^{(2)}_{\lambda \beta}(r/l, \bar{h}')
\]

for \( r \gg a \) lattice constant.\(^{16,17}\)

C. A simple example

In the Ising model, there are two relevant fields, \( h \) and \( t \). Let our reference problem be the ordinary nearest neighbour Ising model. Our comparison problem could be the Ising model of the form

\[
\langle \sigma \rangle = \sum_r h_n' \sigma_r + K_1 \sum_{\text{nearest \ neighbours}} \sigma_r \sigma_{r'} + K_2 \sum_{\text{next nearest \ neighbours}} \sigma_r \sigma_{r'}
\]

(3.9)
Then, let this problem have a critical point at $h^\Pi = 0$, $K_1 = U(K_2)$. Fix the value of $K_2$ and define

$$l^\Pi = -K_1 + U(K_2).$$

Because of the symmetry of the problem, the transformation coefficients in eqns (3.5) are diagonal in $a$ and $b$. Hence the statement of the universality hypothesis is that the free-energy per site for problem (3.9) is

$$f^\Pi(h^\Pi, K_1, K_2) = b_0(K_2) f^I(b_0(K_2) h^\Pi, b_1(K_2) l^\Pi) +\text{ non-singular terms} \quad (3.10)$$

where $f^I(h, t)$ is the free-energy per site for the ordinary Ising model. For another example, the spin–spin correlation function defined by the solution to the problem (3.9) is

$$\langle \sigma(0)\sigma(r) \rangle = g(r, h^\Pi, K_1, K_2).$$

According to universality, this can be expected to take the form

$$g(r, h^\Pi, K_1, K_2) = \left[c_0(K_2)\right]^{-2} g^I(\frac{r}{l(K_2)}, b_0(K_2) h^\Pi, b_1(K_2) l^\Pi) \quad (3.11)$$

for $r$ much bigger than a lattice constant. Here $g^I(r, h, t)$ is the spin–spin correlation function for the pure nearest neighbour case.

Clearly if the statements we have just made are correct, they are very useful in simplifying any discussion of critical phenomena. In particular, they imply that the critical indices are independent of $K_2$.

D. Additional universality statements

The transformation among problems just defined involves the following unknown parameters: $c_{ab}$, $b_{ab}$, $b_0$, $l$. However, one can argue that these parameters are not all independent. In particular, one can use the identities (3.4) to obtain relations among these coefficients. To obtain these assume that

$$\chi_{ab}^\Pi = \frac{\partial^2 f^\Pi}{\partial h_a^\Pi \partial h_b^\Pi} \quad (3.12)$$

is singular and that the leading singularities in this function can be obtained from the singular terms in $f^\Pi$, $m^\Pi$ and $g_{ab}^\Pi$. From (3.4) and (3.6), we find

$$\chi_{ab}^\Pi(h^\Pi) = b_0 \chi_{ab}(h) b_{ab} b_{ab}, \quad (3.13a)$$

if we can neglect the derivatives of the $b$'s with respect to the $h$'s. From eqns (3.7) and (3.8) we can obtain alternative evaluations of $\chi_{ab}$. The results
\[ \chi_s(h')C_{i\ell} = \sum_r \left( \frac{g_{\ell r}(r/l, h')}{\langle s_r h' \rangle} - \langle s_r h \rangle \langle s_h \rangle h' \right) \]
\[ = \mu_{\ell} \lambda_{\ell} \chi_s(h') \]  
(3.13b)

where \( d \) is the dimensionality of the system and
\[ \chi_s(h')C_{i\ell} = \gamma_{i\ell} \chi_s(h)b_{\ell r} \]  
(3.13c)

If we further assume that \( c_{a\ell} \) has an inverse, then the mutual consistency of eqns (3.13) demands
\[ b_{\ell r} C_{i\ell} = \delta_{a\ell}^{\mu} \]
\[ b_0 = l^{-d} \]

In the previous section, we considered a case in which \( b \) and \( c \) were diagonal matrices. In general we shall use Latin indices, \( i, j, k, \ldots \), to describe results in this diagonal representation. In this case
\[ b_{ij} = \delta_{ij} b_i; \quad C_{ij} = \delta_{ij} C_i \]  
(3.16)

with no sum on \( i \) intended. Then, eqn (3.14) reduces to
\[ b_i C_i = \mu_i \]  
(3.17)

IV. The Roots of The Theory: Scaling

A. The irrelevance of the length scale

The physical picture that we have of critical phenomena involves the notion that at the critical point fluctuations at all length scales contribute to the various thermodynamic functions. As a result, one might expect that critical phenomena will show an invariance under the change of length scales
\[ \tau \rightarrow \tau' = \tau/l. \]  
(4.1)

This invariance can be defined by the statement that the transformation (4.1) takes us to a new description of the critical problem which is identical to the old description in the sense that they be within the same universality class.

B. A formal description of the length transformation

Consider a general Ising model problem on some simple lattice, for example, the square lattice of Fig. 4.1a. The most general translationally invariant description of this problem would be to write
\[ \exp - [N f(k)] = \sum_{[\sigma]} \exp [-F(\sigma)] \]
Here the \( S \)'s are all the translationally invariant operators which can be formed from the spins in the problem. Thus, for example \( S_1 \) might be the

![Diagram of a two-dimensional lattice with points inside circles kept in the new lattice of Fig. 4.1b.](image)

**Fig. 4.1a. A two-dimensional lattice. The points inside the circles will be kept in the new lattice of Fig. 4.1b.**

**FIG. 4.1b. The new lattice.**

total magnetization, \( S_3 \) the sum over all products of nearest neighbour spins, etc. Each of these operators is of the form

\[
S_a = \sum_r s_a(r).
\]

To obtain a complete set of operators, we must include the constant term:

\[
S_0 = \sum_r 1 = N
\]

where \( N \) is the number of sites on the lattice.
Now consider a transformation which reduces the number of degrees of freedom in the problem. In formal terms, we can write this transformation as

\[ \exp \left[ \mathcal{F}'(\mu) \right] = \sum_{\{\sigma\}} T\{\mu, \sigma\} \exp \left[ \mathcal{F}(\sigma) \right]. \tag{4.3} \]

We demand that this transformation leave the free energy invariant, i.e.
that

\[ \sum_{\{\mu\}} \exp \left[ \mathcal{F}'(\mu) \right] = \sum_{\{\sigma\}} \exp \left[ \mathcal{F}(\sigma) \right]. \tag{4.4} \]

The condition (4.4) will be true if

\[ \sum_{\{\mu\}} T\{\mu, \sigma\} = 1. \tag{4.5} \]

Let us exhibit one transformation which meets these conditions. The lattice in Fig. 4.1a contains the points,

\[ \varphi = (n, m)a \]

where \( n \) and \( m \) are integers. The lattice in Fig. 4.1b contains only the subset of these points

\[ \varphi' = (n, m)a \quad \text{for} \quad n + m \text{ even}. \]

If there are \( N \) spins in the original lattice, there are \( N/2 \) spins in the new lattice. In fact, the new lattice is exactly the same as the old except that the lattice constant has been increased by the amount

\[ a \rightarrow a' = la \]

\[ l = \sqrt{2}. \tag{4.6} \]

Therefore, two points separated by \( n \) lattice constants in the original lattice are separated by \( n/l \) lattice constants in the new lattice.

If we define new spin variables

\[ \mu_r = \pm 1 \]

at all the points of the new lattice, we have a new Ising model on a transformed lattice with a new length scale, defined as in eqn (4.1).

In particular define

\[ T_r\{\mu, \sigma\} = \prod_{r} (1 + \mu_r \sigma_r)/2. \tag{4.7} \]

Then if \( T \) is \( T_r \), the transformation (4.3) is a summation over the spins at the eliminated sites of the original lattice. For all the sites which lie on the new lattice \( \mu_r, \sigma_r \). Notice that the transformation function (4.7) manifestly obeys eqn (4.4).
More generally, we can write
\[ T\{\mu, \sigma\} = \sum_{\mu' \in \{\pm 1\}} T_2\{\mu, \mu'\} T_1\{\mu', \sigma\} \]  
(4.8)

Here \(\mu'\) and \(\mu\) are spins on the same lattice, i.e. that of Fig. 4.1b. \(T_2\{\mu, \mu'\}\) is any function of \(\mu\)'s and \(\mu''\)'s, which obeys
\[ \sum_{\mu = \pm 1} T_2\{\mu, \mu'\} = 1. \]

The combined transformation (4.8) represents a very general transformation which includes a change in the length scale.

Just as the original free energy function \(\mathcal{F}\{\sigma\}\) can be described by a set of coupling constants \(K_a\), in exactly the same way the new function \(\mathcal{F}'\{\mu\}\) can be written as
\[ \mathcal{F}'(\mu) = \sum_a K'_a S_a(\mu) \]  
(4.9)

Here the \(S_a\)'s are exactly the same function of the spins as defined previously. The arguments of these functions have been changed. If, we recognize that
\[ S_a(\sigma) = \sum_r s_a(r) \]
then we notice that \(S_a(\mu)\) contains only half as many terms as \(S_a(\sigma)\).

In general, we can regard the transformation (4.3) as a change in the values of the coupling constants
\[ K_a \rightarrow K'_a(\vec{K}), \]  
(4.10)
where as indicated the new coupling constants are functions of the old.

C. Fixed points

The basic concept in the Wilson formulation of critical phenomena theory is the idea that for a suitable choice of \(T\) the transformation (4.10) may have a "fixed point". This fixed point is simply a value of the vector of coupling constants \(\vec{K} = K^*\) such that the transformation (3.35) leads to exactly the same value once again,
\[ K'_a(\vec{K}^*) = \vec{K}^*_a \quad \text{for all } a \neq 0 \]  
(4.11)

At this fixed point, the length transformation does not change the problem at all. Hence we identify the fixed point with the critical point.

Notice that eqn (4.11) does not apply to the constant term in the free energy
\[ K_0 S_0 = K_0 \sum_r s_r \]  
(4.12)

Because this constant term does not change any expectation values, it is
allowed to change under the transformation (4.10). In our later work, we shall see that the transformation properties of this term are intimately related to the evaluation of the free energy. For now, we need only note that the "trivial" constant $K_0$ does not enter the evaluation of the $K'_s$ for $\sigma \neq 0$.

Notice also that the numerical value of the vector $K_s^*$ depends upon our choice of $T[\mu, \sigma]$.

D. Deviations from the fixed point

To get physically interesting results from the fixed point theory, we must consider deviations from the fixed point. Write

$$K_s = h_s + K_s^* \quad \text{for } \alpha > 1$$

(4.13)

where $h_s$ measures the deviations. We think of $h_s$ as small. The transformation (4.10) can then be written as

$$K_s \rightarrow K'_s = K'_s(\tilde{K})$$

or

$$K_s^* + h'_s = K'_s(\tilde{K}^* + \tilde{h})$$

(4.14)

when $\tilde{h} = 0$, $h'_s = 0$. Therefore, we can consider $h'_s$ to be at least proportional to $h_s$, i.e.

$$h'_s = b_{sd}h_d$$

(4.15)

where $b_{sd}$ is a function of the $h$'s which reduces to a constant as the $h$'s approach zero. We assume that $b_{sd}$ is analytic in $h$ for small $\tilde{h}$.

A reasonably clear physical picture of what is going on may be obtained if we assume that the transformation (4.15) may be diagonalised by picking a suitable linear combination of the $h_s$'s. In diagonal form (4.15) reads

$$h'_s = b_s h = h_s\Lambda_s.$$  \hspace{1cm} (4.16)

The indices $y_i$ defined in eqn (4.16) are the principal determinants of critical behaviour.

We wish to distinguish three cases:

$$y_i > 0; \ h_i \ \text{is termed a relevant field}$$

$$y_i < 0; \ h_i \ \text{is termed irrelevant}$$

$$y_i = 0; \text{the borderline case.}$$

(4.17)

Imagine that we make a succession of scaling transformations like (4.16). After $n$ such transformations

$$r \rightarrow r' = r/L$$

$$h_i \rightarrow h'_i = h_iL^y \quad \text{where } L = l^n$$


Since the critical behaviour is determined by long-range effects, one can probe the particularly critical effects by looking at the problem after such a transformation. The results of this transformation are that:

- \( k'_i \to 0 \) for irrelevant variables
- \( h'_i \) grows quite large for relevant variables
- \( k'_i = h_i \) for the borderline variables

Hence the irrelevant variables drop out of the critical behaviour; the relevant variables grow; the borderline variables remain fixed.

Thus we can identify the irrelevant variables with those which were described as irrelevant in our previous discussion of universality.

The relevant variables represent relevant deviations from criticality, e.g., \( h \) and \( t \). As we look at longer and longer length scales these variables become more and more important.

The borderline variables play a more subtle role in the theory which we shall discuss below.

E. Scaling results for the free energy

For the time being, set both the borderline and the irrelevant fields equal to zero. We are left with only the relevant fields. We wish to calculate the free energy as a function of these relevant fields. On one hand, we can calculate the free energy as

\[
\exp[N f(h)] = \sum_{\{\sigma\}} \exp[\mathcal{F}^*\{\sigma\} + \sum_i h_i S_i(\sigma)].
\]  
(4.18)

On the other hand, we can calculate the free energy as

\[
\exp[N f(h)] = \sum_{\{\mu\}} T\{\mu, \sigma\} \exp[\mathcal{F}^*\{\sigma\} + \sum_i h_i S_i(\sigma)]
= \sum_{\{\mu\}} \exp[\mathcal{F}^*\{\mu\} + h'_i S'_i(\mu) + f_0(h)N']
\]  
(4.19)

where \( N' \) is the number of sites in the new lattice. The \( h'_i \) are the transformed field variables. The term in \( f_0(h) \) is a constant term in the free energy, independent of all the variables \( \mu(r') \). Because the transformation only involves summations over small regions of the original lattice, we argue that \( f_0(h) \) is an analytic function of the \( h'_i \)'s near \( h_i = 0 \). A further evaluation of the sum on the right-hand side of eqn (4.19) gives

\[
\exp[N f(h)] = \exp\{N'[f_0(h) + f(h')]\}
\]  
(4.20)

since the sum over \( \mu \)'s in (4.19) is of exactly the same form as the sum over...
\(o's\) in (4.18). Since \(N' = N/t^2\), we see that
\[
f(h) = \frac{1}{2^t} \left[ f_o(h) + f(h') \right]
\]
(4.21)
\[
= \frac{1}{2^t} \left[ f_o(h) + f(b_r h) \right].
\]

Now take a particular case. Assume that there are two thermodynamically significant variables \(h\) and \(t\). Write \(b_r = p_r = p\). Equation (4.21) reads
\[
f(h, t) = \frac{1}{2^t} \left[ f_o(h, t) + f(p^r h, p^t t) \right].
\]

A solution to this equation is
\[
f(h, t) = \lambda^{d(1-\gamma)} y_y(h/t^\gamma y_f) + \text{regular function of } h \text{ and } t.
\]
(4.22)

Using this form of the solution we can identify the variables \(\alpha, \beta, \gamma, \gamma', \delta\) in terms of the pair of variables \(d/y_i\) and \(y_i/y_i\) as
\[
2 - \alpha = d/y_i,
\]
\[
\beta = \frac{d}{y_i} - y_i/y_i,
\]
\[
\gamma = \frac{d}{y_i} - 2y_i/y_i,
\]
\[
1 + \delta^{-1} = d/y_i
\]
(4.23)

so that we find the relationships
\[
2 - \alpha = \gamma + 2\beta = \gamma' + 2\beta = \beta(\delta + 1).
\]
(4.24)

The data in Table 1.1 support these assertions.

F. A simple example of the theory

As the simplest example of the theory consider the nearest neighbour, one-dimensional Ising model:
\[
\exp(\mathcal{F}(\sigma)) = \prod \exp(K(\sigma_i, \sigma_{i+1} + h\sigma_i))
\]
(4.25)

This model has a kind of critical point at \(h = 0\) and zero temperature \((K = \infty)\). We describe the solution in terms of the two variables \(h\) and
\[
t = \exp - K.
\]
(4.26)

In terms of these variables, the solution to this problem in the limit \(t \to 0\), \(h \to 0\) is given by
\[
f(h, t) = - \ln t + \sqrt{(t^4 + h^2)}
\]
(4.27)
1. Scaling, universality and operator algebras

\[ \langle \sigma \rangle = \frac{h}{\sqrt{(h^2 + r^4)}} \]  
\[ \langle \sigma_i \sigma_{i+1} \rangle = \langle \sigma \rangle^2 + [1 - \langle \sigma \rangle^2] \exp - r / \xi \]  
(4.29)

where

\[ \xi^{-1} = 2 \sqrt{(r^4 + h^2)}. \]  
(4.30)

Now obtain the results of the renormalization method for this problem. Choose

\[ \mu_i = \sigma_{2j} \]

and sum expression (4.25) over all \( \sigma_i \) for odd values of \( i \). This particular choice of renormalization procedure corresponds to picking

\[ l = 2 \]  
(4.31)

and \( T \) equal to the \( T_i \) defined by eqn (4.7). Then we have

\[
\exp(\mathcal{S}(\mu)) = \sum_{\sigma_{2j+1} = \pm 1} \prod_j \left( \exp \left( h \sigma_{2j} + \sigma_{2j+2} \right) \right) 
\times \exp \left( h(\sigma_{2j} + \sigma_{2j+2}/2) \right) 
\times 2 \cosh \left[ h + K(\mu_j + \mu_{j+1}) \right].
\]

Notice that this result can exactly be written in the form

\[
\exp(\mathcal{S}(\mu)) = \prod_j \left( f_0 + K \mu_j \mu_{j+1} + (h/2)(\mu_j + \mu_{j+1}) \right)
\]

where

\[ \frac{h}{2}(\mu_j + \mu_{j+1}) + f_0 + K \mu_j \mu_{j+1} = (h/2)(\mu_j + \mu_{j+1}) \]

+ \ln 2 \cosh \left[ h + K(\mu_j + \mu_{j+1}) \right].

so that

\[ h' = h + \frac{1}{2} \ln \frac{\cosh (2K + h)}{\cosh (2K - h)} \]

\[ f_0 = \ln 2 + \frac{1}{4} \ln \cosh (2K + h) \cosh (2K - h) + \frac{1}{2} \ln \cosh h \]

\[ K' = \frac{1}{4} \ln \cosh (2K + h) \cosh (2K - h) - \frac{1}{2} \ln \cosh h. \]

In the limit as \( K \to \infty \) \( + h \to 0 \) these results reduce to

\[ h' = 2h \]

\[ f_0 = K + \frac{\ln 2}{2} = \ln t + \ln \frac{2}{2} \]  
(4.32)

\[ t' = \sqrt{2t} \]
As a result, eqn (4.21) reads

\[ f(h, t) = \frac{1}{2} \left[ -1 \ln t + \frac{\ln 2^2}{2} \right] + \frac{1}{t} f(p^0 h, P h^t) \]  

(4.33)

with \( y_h = 1, y_t = \frac{1}{2}. \) The solution of (4.33) is

\[ f(h, t) = -\ln t + \frac{t^2 f(h/t^2)}{2} \]  

(4.34)

which is exactly of the form (4.27). We identify the first term in (4.34) as the analogue of the "regular" term in critical phenomena theory and the second term as the analogue of the "scaling" part.

V. Scale Transformations for Correlation Functions

A. General discussion

The scaling transformation, as described in the previous section, can be described in the following fashion. A length transformation

\[ r \rightarrow r' = r/l \]  

(5.1)

converts the critical phenomena problem into another, essentially similar problem in which the basic field transform according to

\[ h_x \rightarrow h'_x = b_x h_x \]  

(5.2)

where \( b_x \) are power series expandable in \( \vec{h} \) for small \( h. \) Compare this statement to the definitions of the universality in Section III. B. It seems reasonable to believe that the transformation defining the change in length scale is a particular example of a transformation which takes us from one problem to another in the same universality class. If this be true then the scaling theory should yield restrictions upon the correlation functions quite analogous to the restrictions implied by the universality arguments.

Another way of making the same point is to say that the scale transformation idea is based upon the scale invariance of the fluctuations in the problem. These fluctuations are best studied in the correlation functions. Hence the scale-transformation theory should be most fully reflected in the behaviour of correlation functions.

B. Formal theory

The most direct way of studying this behaviour is to consider the operators \( S_x(\sigma) \) to be sums of local operators

\[ S_x(\sigma) = \sum_r s_x(\sigma, r). \]  

(5.3)
1. Scaling, universality and operator algebras

Each $s_\alpha(\sigma, r)$ is expected to depend upon only the $\sigma(r')$ for $r'$ in the immediate neighbourhood of $r$. Now consider a generalization of the transformation (4.3)

$$\exp(\mathcal{F}\{\mu\}) = \exp[\mathcal{F}^*\{\mu\} + \sum_{x=0}^{\infty} h_x S_x\{\mu\}]$$

$$= \sum_{\{\sigma\}} T_{\{\mu, \sigma\}} \exp[\mathcal{F}^*\{\sigma\} + \sum_{x=0}^{\infty} h_x S_x\{\sigma\}]$$

to the case in which $h_x$ depends upon $r$

$$\exp[\mathcal{F}^*\{\mu\} + \sum_{r'} \sum_{x=0}^{\infty} h_x(r') s_x(\{\mu\}, r')]$$

$$= \sum_{\{\sigma\}} T_{\{\mu, \sigma\}} \exp[\mathcal{F}^*\{\sigma\} + \sum_{x=0}^{\infty} h_x(r) s_x(\{\sigma\}, r)] \quad (5.4)$$

$$= \sum_{\{\sigma\}} T_{\{\mu, \sigma\}} \exp(\mathcal{F}\{\sigma\}).$$

Here $s_x(\{\mu\}, r')$ is exactly the same function of the neighbouring $\mu$'s as $s_x(\{\sigma\}, r)$ is one of the neighbouring $\sigma$'s.

We shall apply eqn (5.4) to the case in which $h_x(r)$ varies slowly in $r$. When $h_x$ was independent of $r$ and small, we asserted that

$$h_x' = b_x h_x \quad (5.5)$$

Now make the analogous assertion for the use of slow spacial variation.

$$h_x'(r') = b_{x\beta} h_{\beta}(r) \quad \text{for } r' \approx \tau/l$$

Or, if we wish to be more careful we can write

$$h_x'(r') = \sum_{r} b_{x\beta}(r', r) h_{\beta}(r) \quad (5.6)$$

with

$$b_{x\beta}(r', r) \approx 0 \text{ unless } (\tau' - \tau/l) \approx \text{ a few lattice constants} \quad (5.7)$$

By considering the case $h_\beta(r)$ independent of $r$ and comparing (5.6) with (5.5), we find

$$\sum_{r} b_{x\beta}(r', r) = b_{x\beta} \quad (5.8a)$$

On the other hand, sum eqn (5.6) over $r'$, when $h_\beta(r)$ is independent of $r'$.

The equation becomes:

$$N' h_x = \sum_{r} [\sum_{r'} b_{x\beta}(r', r)] h_{\beta} = N(\sum_{r} b_{x\beta}(r', r) h_{\beta}).$$
\[ l^{-4}h'_s = \sum_{r'} b_{s\mu}(r', r)h_{\mu} \]

consequently

\[ \sum_{r'} b_{s\mu}(r', r) = l^{-4}b_{s\mu}. \quad (5.8b) \]

The difference between (5.8a) and (5.8b) arises precisely because there are more terms in a sum over \( r \) than in a sum over \( r' \).

Now we are ready to calculate averages and correlation functions.

\[
\langle s_s(r) \rangle_h = \frac{\left[ \delta / \delta h_{\mu}(r) \right] \Sigma_{\{\sigma\}} \exp(\mathcal{F}\{\sigma\})}{\Sigma_{\{\sigma\}} \exp(\mathcal{F}\{\sigma\})}
\]

\[
\langle s_s(r_1)s_s(r_2) \rangle_h = \frac{\left[ \delta / \delta h_{\mu}(r_1) \right] \left[ \delta / \delta h_{\mu}(r_2) \right] \Sigma_{\{\sigma\}} \exp(\mathcal{F}\{\sigma\})}{\Sigma_{\{\sigma\}} \exp(\mathcal{F}\{\sigma\})}
\]

but,

\[ \sum_{\{\mu\}} \exp(\mathcal{F}\{\mu\}) = \sum_{\{\mu\}} \exp(\mathcal{F}'\{\mu\}) \]

consequently,

\[
\langle s_s(r) \rangle_h = \frac{\left[ \delta / \delta h_{\mu}(r) \right] \Sigma_{\{\mu\}} \exp(\mathcal{F}'\{\mu\})}{\Sigma_{\{\sigma\}} \exp(\mathcal{F}'\{\sigma\})}
\]

If we take \( b_{s\mu}(r', r) \) in eqn (5.6) to be independent of \( h_{\mu} \), as will be true for sufficiently small \( h_{\mu} \), then eqn (5.9) becomes

\[ \langle s_s(r) \rangle_h = \sum_{r'} \frac{\Sigma_{\{\mu\}} s_s(\{\mu\}, r')b_{s\mu}(r', r) \exp(\mathcal{F}'\{\mu\})}{\Sigma_{\{\mu\}} \exp(\mathcal{F}'\{\mu\})} \]

The left-hand side of this equation is \( M_s(h) \). The right-hand side is

\[ \sum_r M_s(h')b_{s\mu}(r', r) \]

With the aid of (5.8b), we find

\[ M_s(h) = M_{s}(h')l^{-4}b_{s\mu}. \quad (5.10) \]

An analogous derivation applied to the correlation function yields

\[
G_{s\mu}(\vec{r}_1 - \vec{r}_2, h) = \langle s_s(r_1)s_s(r_2) \rangle_h = \\
\sum_{r_1, r_2} \langle s_s(r_1)s_s(r_2) \rangle_h b_{s\mu}(r_1, r_1)b_{s\mu}(r_2, r_2) + \sum_{r_2} M_s(h') \\
\times \frac{\delta}{\delta h_s(r)} b_{s\mu}(r_2, r_2) \quad (5.11)
\]
1. Scaling, universality and operator algebras

Equation (5.11) simplifies greatly if

$$\left| \vec{r}_1 - \vec{r}_2 \right| \gg \text{a lattice constant.}$$

In this case the correlation function is sufficiently slowly varying so that it can be taken outside the first sum. Also, the coefficient \(b_{\alpha\beta}(r')\) should only depend upon \(h'(r)\) in its immediate neighbourhood. Consequently the second term can be dropped. As a result, (5.11) simplifies to

$$G_{\alpha\beta}(r_1 - r_2, h) = G_{\mu\nu} \left( \frac{r_1 - r_2}{l}, h' \right) b_{\mu\alpha} b_{\nu\beta} l^{-2d}$$

for \(|r_1 - r_2| \gg \text{a lattice constant.} \quad (5.12)$$

Equations (5.10) and (5.12) are the objects of our derivations. They are a direct result of the extension of the transformation theory to slowly varying \(h'(r)\), with the assumption that the transformations are quasi-local in \(\vec{r}\).

C. Connections with universality

To make a connection with the universality concept define a matrix \(c_{\alpha\beta}\) by

$$b_{\alpha\gamma} c_{\gamma\beta} l^{-d} = 1; \quad (5.13)$$

of course this definition will fail if \(b\) is not invertible. Then define

$$s'(r') = s_\delta(\{\sigma\}, r)c_{\rho\delta} \quad (5.14)$$

for all \(r' = r/l\) for which \(r\) lies on the new lattice. According to eqns (5.10) and (5.12), all the critical point equations will remain true if we make the transformations

$$h \rightarrow h'_\alpha = b_{\alpha\beta} h_\beta$$
$$r \rightarrow r' = r/l$$
$$s_\delta(r) \rightarrow s'_\delta(r') = s_\delta(r)c_{\rho\delta} \quad (5.15)$$

Equations (5.15) are exactly of the same form as our universality transformation. Hence one can describe scaling theory as equivalent to the statement that the universality hypothesis applies to scale transformations.

D. Consequences

To obtain the physical consequences of the scaling theory consider a representation in which \(b\) and \(c\) are diagonal

$$b_{ij} = \delta_{ij} l^\rho$$
$$c_{ij} = \delta_{ij} l^{\nu}$$

$$x_i + y_i = d. \quad (5.16)$$
The last relation follows in virtue of eqn (5.13). Then eqn (5.10) implies that

\[ M_i(h_1, h_2, \ldots) = l^{-x_i} M_i(|l| h_1, |l| h_2, \ldots) \]

and consequently

\[ M_i(h_1, h_2, \ldots) = h_i^{x_i/y_i} M_i \left( \frac{h_2}{h_i^{x_i/y_i}}, \frac{h_3}{h_i^{x_i/y_i}}, \ldots \right) \]

In the special case in which \( h_1 \) is \( h, h_2 \) is \( t \) and there are no other variables in question, we find

\[ M_i(h, t) = \langle \sigma \rangle \]

\[ = h^{x_i/y_i} M_i \left( \frac{t}{h^{x_i/y_i}} \right) \]

or equivalently

\[ \langle \sigma \rangle = t^{x_i/y_i} M_i \left( \frac{h}{t^{x_i/y_i}} \right) \]

(5.17)

If \( y_i + x_i = d_i \), this equation contains no new information beyond that in eqn (4.22).

However, the correlation function information is new. From eqn (5.12)

\[ G_{ij}(r, h, t) = G_{ij}(|r| h, |r| t) \]

Hence

\[ G_{ij}(r, h, t) = \frac{1}{r^{x_i+x_j}} g \left( r^{1/y_i} \frac{h}{t^{x_i/y_i}} \right) \]

(5.18)

Equation (5.18) enables us to identify the value of \( \xi \) at \( h = 0 \) as proportional to \( r^{-1/y_i} \). Hence \( \nu \) is given by

\[ \nu = 1/y_i \]

(5.19)

The additional information about critical indices contained in eqn (5.19) is derived by comparing the first of eqns (4.23) with (5.19). We then see that

\[ d \nu = 2 - \alpha \]

(5.20)

Furthermore, eqn (5.18) implies that \( x_h \) and \( x_i \) are directly observable since

\[ g(r, h = 0, t = 0) = \frac{1}{r^{x_h}} \text{ for } r \gg \alpha \]

(5.21)
1. Scaling, universality and operator algebras

\[ \langle \mu(0)\mu(r) \rangle \sim \frac{1}{r^{2\Lambda}} \]

These observable quantities are derivable from thermodynamic data and the last of eqns (5.16), i.e.

\[ x_h + y_h = d \]
\[ x_i + y_i = d \]

hence,

\[ x_i = d - \frac{1}{\nu} \]
\[ = d \left(1 - \frac{1}{2 - \alpha} \right) = d \frac{1 - \alpha}{2 - \alpha} \tag{5.22} \]

while from (4.23)

\[ x_h = d \left(1 - \frac{\delta}{\delta + 1} \right) = \frac{d}{\delta + 1} \tag{5.23} \]

Equations (5.20), (5.22), and (5.23) are exactly satisfied for the two-dimensional Ising model. However, the available evidence suggests that (5.20) is wrong for the three-dimensional Ising model.

References

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11. See, for example, Ref. 5 and J. M. H. Levelt-Sengers in "Proceedings of the Enrico Fermi Summer School of Physics, Varenna, 1970” (M. S. Green ed.),
16. The two scale factor universality has been verified by some series calculation.
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