The concept of scaling and dimensional analysis has been extensively used in hydrodynamics, and most especially in the study of turbulence. But, in fact, scaling is used in many different ways in this subject.

**DIMENSIONAL ANALYSIS**

At its simplest, scaling describes the elimination of dimensional quantities and the isolation of dimensionless combinations which then describe the situation in question. Let me describe the simplest example: the behavior of the Navier Stokes equation

\[ u_t + (u \cdot \nabla)u = -\frac{\nabla p}{\rho} + \nu \nabla^2 u \]

Since the kinematic viscosity is quite small for most fluids (for example being \(1.0 \times 10^{-6} m^2/sec\) for water and \(1.5 \times 10^{-5} m^2/sec\) for air) the Reynolds number is often quite high.

One uses this kind of scaling by saying that for a fixed geometry the dimensionless flow only depends upon the Reynolds number. Thus, for example, in the picture book by Van Dyke\(^1\), many examples are given of flow past a sphere. Neither the velocity at infinity, \(U\), nor the radius of the sphere, \(L\), are specified but only the Reynolds number. For low values of \(Re\) the flow is laminar, and then as \(Re\) is increased the flow gets more complex and unsteady, until at still larger values it can be said to be truly turbulent.

In the study of convective flow, the
\[ \nabla \cdot u = 0 \quad (1) \]

in a situation in which one is looking at a flow characterized by a typical magnitude of the velocity, \( U \), and a typical length scale \( L \). If one then uses the dimensionless quantities for velocity \( v = u/U \), for length \( R = r/L \), and for time \( T = tU/L \) then the Navier-Stokes equation becomes

\[
\begin{align*}
v_T + (v \cdot \nabla)v &= -\nabla P + \text{Re}\nabla^2 v \\
\nabla \cdot v &= 0
\end{align*}
\quad (2)
\]

In Eq. (2) the gradient is a derivative with respect to the rescaled space variable, \( R \), while \( P \) is a rescaled pressure defined to keep the velocity while the Reynolds number, \( \text{Re} \), is defined by

\[ \text{Re} = \frac{UL}{\nu} \quad (3) \]

Boussinesq equations are also converted into dimensionless form. The physical temperature variable, \( T \), is rescaled by a typical temperature difference in the system, \( \Delta \), to get the dimensionless combination \( \theta = T/\Delta \). Then the relevant quantities obey rescaled equations in which there is two dimensionless parameters, the Rayleigh number:

\[ Ra = \frac{g\alpha\Delta L^3}{\nu} \quad (4) \]

and the Prandtl number

\[ Pr = \frac{\nu}{\kappa} \quad (5) \]

Here \( \kappa \) is the thermal diffusivity, \( g \) the acceleration of gravity, and \( \alpha \) the thermal expansion coefficient of the fluid. The Rayleigh number measures how hard the fluid is being 'pushed' so that turbulence occurs for large values of \( Ra \).

**SIMILARITY SOLUTIONS**

This 'naive' dimensional analysis is only the simplest example of the kind of scaling analysis used in the field. Often the scaling is based upon a relatively sophisticated view of similarity solutions. The simplest and most classical analysis of this kind is due to Blasius and treats the flow past a flat plate.

To see the structure of the result take the curl of the Navier-Stokes equation to find

\[
(u \cdot \nabla)\nabla \times u = \nu \nabla^2 \nabla \times u \quad (6)
\]

The Blasius solution is based upon the idea that there is a characteristic scale of the \( y \) coordinate, which varies with \( \tau \), the dis-

![Blasius Flow Past Plate](image.png)

**Figure 1**: Blasius Figure. This figure shows first the geometry of the situation described by Blasius and then the form of the velocity profile achieved.
tance along the plate measured from its beginning. Therefore, every function of $y$ will depend upon $y$ in the form of functions of $y/Y(x)$ where $Y(x)$ is the characteristic scale of $y$. The scale of $x$ is simply the distance from the leading edge of the plate. We look far downstream, where the scale of $Y$ is much smaller than the scale of $x$. In this limit, one can neglect the $(\partial/\partial x)^2$ term in $\nabla^2$ in comparison with the $(\partial/\partial y)^2$ term. Thence (6) reduces to

\[ (u \cdot \nabla) \nabla \times u = \nu \frac{\partial^2}{\partial y^2} \nabla \times u \quad (7) \]

with the additional condition that the divergence of the velocity vanishes.

To an order of magnitude one can estimate the size of a $y$-derivative as being proportional to $Y(x)^{-1}$, while an $x$-derivative is of the order of the inverse $x$-scale $x^{-1}$. A typical value of $u_x$ is just $U$. Thus, if the two terms in Eq (7) are to balance out, we must have that, to an order of magnitude:

\[ \frac{U}{x} \sim \frac{\nu}{Y(x)^2} \]

so that the $y$-scale is

\[ Y(x) = \left( \frac{\nu x}{U} \right)^{1/2} \quad (8) \]

Using this idea, one can in fact, construct a solution of (7) assuming that the $x$-component of the velocity has the form

\[ u_x = U \Psi \left( \frac{y}{Y(x)} \right) \quad (9) \]

with $\Psi$ obeying suitable boundary conditions.

The physical idea upon which this is all based is that the flow is the same for all $x$ except for the change of scale, represented by the $Y(x)$. Notice that the answer can be represented by simple power laws, for example that $Y(x)$ is proportional to $x^{1/2}$. We see here that the simple power laws are an outcome of the idea that as one goes to larger $x$ nothing changes except the scale of $y$.

**INERTIAL RANGE SCALING**

There is a third use of scaling in turbulence theory based upon the notions of Kolmogorov. In 1941[21], he imagined that turbulence could be described as the flow of energy from large scales to small. He visualized the energy to be injected at a rate $\varepsilon$ as a large kind of eddy. Then the

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