Some Critical Properties of the Eight-Vertex Model

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The eight-vertex model solved by Baxter is shown to be equivalent to two Ising models with nearest-neighbor coupling interacting with one another via a four-spin coupling term. The critical properties of the model in the weak-coupling limit are in agreement with the scaling hypothesis. In this limit where \( \alpha \to 0 \), the critical indices obey \( \gamma/\nu = \nu/\nu_{0} = 1 - \frac{1}{2} \alpha \), with the subscripts zero denoting the index values for the ordinary two-dimensional Ising model.

In a recent publication, Baxter has found the free energy for the eight-vertex problem and shown that \( \alpha \) is a continuous function of the interaction constants. This continuous variation of a critical index contradicts the hypothesis of smoothness or universality often postulated for near-critical problems.

One way of seeing the source of this behavior is to rephrase the eight-vertex problem as an Ising model. Imagine a spin placed at the interstitial points of the lattice as in Fig. 1. An arrow to the right (or upward) corresponds to the case in which the adjacent spins are parallel; a leftward or downward arrow makes the adjacent spins antiparallel.

Then, the four combinatorical factors \( a, b, c, \) and \( d \) corresponding to the vertices shown can all be represented by a factor in the partition function

\[
A e^{x_{1} q_{1} + x_{2} q_{2} + \lambda q_{3} q_{4}}
\]

and we obtain the complete partition function

\[
\sum_{q_{i},q_{4} \in \{0,1\}} \prod_{j,k} \left[ e^{x_{1} q_{j,i} q_{j,i+1} + x_{2} q_{j,i} q_{j,i+1} + \lambda q_{j,i} q_{j,i+1} q_{j,i+1} q_{j,i+1}} + e^{x_{1} q_{j,i} q_{j,i+1} + x_{2} q_{j,i} q_{j,i+1} - \lambda q_{j,i} q_{j,i+1} q_{j,i+1} q_{j,i+1}} \right],
\]

in which next-nearest-neighbor spins are coupled by interaction constants \( K^* \) depending upon the direction of the diagonal. The factor \( \lambda \) couples all four spins. The precise connection is that

\[
\begin{align*}
    \alpha &= A e^{x_{1} q_{j,i} q_{j,i+1}}, \\
    \beta &= A e^{-x_{2} q_{j,i} q_{j,i+1}}, \\
    \gamma &= A e^{x_{1} q_{j,i} q_{j,i+1}}, \\
    \delta &= A e^{-x_{2} q_{j,i} q_{j,i+1}}.
\end{align*}
\]

The constant \( A \) does not, of course, enter into the critical properties.

The Baxter solution shows that this Ising-type problem has a new kind of singularity at the critical point, namely, one in which the singularity in the specific heat as \( \epsilon = (\beta + c - d - a)/a \) goes to zero is of the form \( \epsilon^{\alpha} \) with \( \alpha \) being a function of the parameters, namely,

\[
\sin \frac{\pi \alpha}{4(1 - \frac{1}{2} \alpha)} = \tanh 2 \lambda.
\]

This result seems at first to contradict the smoothness hypothesis which suggests that critical indices should not change their value unless there is a symmetry change. However, this eight-vertex model certainly has a different set of symmetries than the usual two-dimensional Ising model. Notice that at \( \lambda = 0 \), the lattice with \( j + k = \) (even integer) does not interact with the lattice with \( j + k = \) (odd integer). Even at \( \lambda \neq 0 \) for \( T > T_{c} \), i.e., \( \epsilon > 0 \), the spins on these two sublattices are uncorrelated. Therefore, the Ising form of the eight-vertex model can be viewed as having two lattices with "independent" ferromagnetic transitions which occur at exactly the same temperature. The coupling between these two lattices is of the form

\[
\lambda \sum_{r} u_{r}; \quad u_{r} = g_{r}^{(1)} g_{r}^{(2)},
\]
FIG. 1. The correspondence between the Ising-spin configurations, the eight-vertex configurations, and the Boltzmann factors \(a, b, c, d\).

where \(S^{(a)}\) and \(S^{(b)}\) are the energy densities on the two sublattices. This kind of coupling leaves the spontaneous magnetizations on the two lattices free to point in either the same or in opposite directions. Since this two-sublattice symmetry is very different from that of the usual Ising model, it is not surprising that the critical indices of the Baxter solution are, in general, different from those of the Onsager solution.

A second unexpected feature of the solution is that \(a\) varies continuously with \(\lambda\). The scaling hypothesis usually rules out this idea as is shown in the discussion of Ref. 3. However, there is one special case in which the scaling idea does permit the continuous variation of critical indices—if there is a term in the Hamiltonian of the form of \(\lambda \sigma^a \bar{u}_t\) and \(\bar{u}_t\) scales as \(1/r^d\) (\(d\) denotes the dimension of the lattice).

To see why this particular scaling is so significant, recall the definition of scaling: In the critical region, the phase transition is supposed to be described by fluctuating local quantities, e.g., the magnetization density and the energy density, which we can write as \(O_\alpha(r)\). The \(\alpha\) distinguishes among different quantities. Let \(O\) be a product of \(n\) different quantities of this type,

\[
O = \prod_{i=1}^{n} O_{\alpha_i}(r_i) ,
\]

and take each pair of operators in the product to be separated by a distance \(|\Gamma_i - \Gamma_j|\) of the order of magnitude of \(R\), with \(R\) much greater than a lattice constant and much smaller than a coherence length. Then the statement \(O_\alpha(r)\) scales as \(1/r^d\) means precisely\(^6\) that

\[
\langle O \rangle_{x, \lambda} \sim \frac{1}{R^x}, \quad x = \sum_{j=1}^{n} x_{\alpha_j}
\]

for \(n \geq 2\).

Here the \(x_{\alpha_j}\)s are critical indices which describe the behavior of the fluctuating variables. For the ordinary two-dimensional Ising model, the magnetization has an index \(x_\sigma = \frac{1}{4}\), and the energy density \(x_\varepsilon = 1\).

If these indices vary with \(\lambda\), then the derivative of \(\langle O \rangle_{x, \lambda}\) contains a term like \(R^x \ln R\), in particular,

\[
\frac{\partial \langle O \rangle_{x, \lambda}}{\partial \lambda} = -\langle O \rangle_{x, \lambda} \sum_{j=1}^{n} \left( \frac{\partial x_{\alpha_j}}{\partial \lambda} \right) \ln R + \ldots ,
\]

where the \(\ldots\) represents terms which are not logarithmic in \(R\). Therefore, these logarithmic terms are signals of continuously varying critical indices.

To see how this logarithm can arise, notice that if \(\lambda\) is conjugate to \(a_\sigma\), which contains a term \(\bar{u}_t\), then

\[
\frac{\partial}{\partial \lambda} \langle O \rangle_{x, \lambda} = \sum_{\Gamma} \langle O \bar{u}_t \rangle_{x, \lambda} + \ldots .
\]

According to the operator algebra concept, when a product of two operators which are relatively close to one another appears inside a correlation function, their product may be approximately replaced according to

\[
O_a(r_1)O_b(r_2) = \sum_{\Gamma} A_{ab, \gamma}(\Gamma_1 - \Gamma_2)O_\gamma(\frac{1}{2}(r_1 + r_2)) ,
\]

where, according to scaling,

\[
A_{ab, \gamma} = a_{ab, \gamma}(\Gamma_1 - \Gamma_2)/(r_1 - r_2) \left| \sum_{\Gamma} a_{ab, \gamma}(\Gamma_1 - \Gamma_2) \right| \left| r_1 - r_2 \right| \left| \sum_{\Gamma} a_{ab, \gamma}(\Gamma_1 - \Gamma_2) \right|
\]

for separations \(|r_1 - r_2|\) large in comparison to the lattice constant. In the particular case in which \(O_b\) is \(\bar{u}_t\), which scales as \(1/r^d\), then the product in (6) contains a term of the form

\[
O_a(r_1)\bar{u}(r) = \frac{a_{\sigma a}(\frac{1}{2}(r_1 + r))}{\left| r_1 - r \right|} + \ldots ,
\]

when \(O_a\) and \(\bar{u}\) are scalars under rotation. Here \(a_\sigma\) is, of course, the particular coefficient which appears in the reduction formula (9) when \(\alpha = \gamma\) and \(O_b = \bar{u}\).

As a result, the sum in (7) contains a succession of terms

\[
\frac{\partial}{\partial \lambda} \langle O \rangle_{x, \lambda} = \sum_{\Gamma} \sum_{r, \gamma} \left| r - \gamma \right| a_{\gamma}(\Gamma_1 - \Gamma_2) \langle O \rangle_{x, \lambda} + \ldots
\]

which corresponds to \(|r - \gamma|\) being much smaller than the average separation distance \(|r_1 - r_2| \sim R\). Here the \(\ldots\) include all terms in which all separations are at least of order \(R\). The logarithms then appear in \(\sum_{\Gamma}\). In two dimensions one obtains

\[
\sum_{r, \gamma} 1/r^2 \approx 2\pi \ln(R/d_0) .
\]

When Eqs. (11) and (12) are combined, a set of logarithms appears in the derivative. A comparison with Eq. (6) then shows that

\[
\frac{\partial x_{\alpha}}{\partial \lambda} = -2\pi a_\sigma .
\]

We apply this result to the model solved by Baxter
for the particular case $\lambda = 0$. At $\lambda = 0$, the operator
\[
\hat{u}_r = \delta \hat{S}^{(1)} \delta \hat{S}^{(3)}
\]
(14)
does indeed scale as $1/r^2$ if $\delta \hat{S}^{(1)}$ and $\delta \hat{S}^{(3)}$ are the deviations of the energy on the two sublattices from their critical values. To see this, calculate $\langle \hat{u}_r \hat{u}_0 \rangle$ at $\lambda = 0$ and at the critical point. At $\lambda = 0$, the two sublattices are independent so that
\[
\langle \hat{u}_r \hat{u}_0 \rangle = \langle \delta \hat{S}^{(1)} \delta \hat{S}^{(1)} \rangle^2.
\]
However, the statement that $x = 1$ at $\lambda = 0$ implies that at criticality for large $r$
\[
\langle \delta \hat{S}^{(1)} \delta \hat{S}^{(1)} \rangle = q/2\pi r^2.
\]
From Ref. 7 we obtain $q = 4/\pi$. (Note that nearest neighbors in the sublattices are separated by $\sqrt{2}$.)

The correlation function on the left-hand side of (15) is then $(q/2\pi r^2)^2$ and, consequently, $\hat{u}_r$ scales as $1/r^2$.

Because $\delta \hat{u}_r$ has this special value of the scaling index, the critical phenomena theory indicates that the critical indices can vary continuously in $\lambda$.

Conversely, if there is no operator with index $d$, then there can be no continuous variation of the critical indices.

To find the first variation in the critical index for the energy, calculate
\[
\delta \hat{S}^{(1)} \delta \hat{u}_r (q^2) = \delta \hat{S}^{(1)} \delta \hat{S}^{(1)} \delta \hat{S}^{(3)}.
\]
(17)

According to Eq. (17), as $r_1$ approaches $r_\ast$ at $\lambda = 0$, the product of the energy fluctuations on lattice (1) can be replaced by a constant divided by $|r_1 - r_\ast|^2$.

In particular,
\[
\delta \hat{S}^{(1)} \delta \hat{u}_r (q^2) \sim \frac{q}{2\pi |r_1 - r_\ast|^2} \delta \hat{S}^{(1)}.
\]
(18)

Note, however, that this result is not of the form (10), needed to reach Eq. (13). To achieve the form (10), we consider the combinations
\[
\delta \hat{S}^r = \delta \hat{S}^{(1)} \pm \delta \hat{S}^{(3)}
\]
(19)
which have a simple symmetry under the interchange of the two lattices. Equations (18) and (19) give
\[
\delta \hat{S}^r \delta \hat{u}_r (q^2) = \pm \frac{q}{2\pi |r_1 - r_\ast|^2} \delta \hat{S}^r.
\]
(20)

when $r_1$ and $r_2$ are relatively close together compared to all other distances but $|r_1 - r_\ast|$ is large compared to a lattice constant. It now follows that $\delta \hat{S}^r$ scales as $1/r^\beta$ with Eq. (10) giving
\[
\frac{dx}{d\lambda} = \mp q.
\]
Since the scaling index is 1 at $\lambda = 0$, we find that for small $\lambda$
\[
x = 1 - \lambda q.
\]
A similar argument applied to $\sigma_r^{(1)}$ indicates that at $\lambda = 0$, $dx_d/d\lambda = 0$ so that to first order in $\lambda$
\[
x_d = \frac{1}{2} q.
\]

To derive this result, notice that for $r_1$ close to $r_2$,
\[
\sigma_r^{(1)} \sigma_{r_2} = \sigma_r^{(1)} \delta \hat{S}^{(1)} \delta \hat{S}^{(3)}
\]
contains no term which is like $\sigma_r^{(1)}$ since this expression contains a reference to fluctuations on lattice 2. Hence the coefficient $q$ in Eq. (13) vanishes. It follows that $\eta = 2x_\ast$ does not change to first order in $\lambda$.

From these results and scaling theory, we can predict all critical indices to first order in $\lambda$. For example, the deviation of energy from criticality contains a singular term of the form
\[
\delta E \sim k^\beta (k-\xi)\rho
\]
where $\xi$ is the correlation length, since $x_\ast$ is the index which goes with the energy. Also the free energy contains a singular term like
\[
\delta F \sim k^\beta \rho
\]
Since $k \sim \epsilon$, $\delta F \sim \epsilon^\beta \rho$, and $\delta E \sim \epsilon^{1-\beta}$, we find
\[
(2 - x_\ast) \nu = 1
\]
or
\[
\nu = 1 - q\lambda,
\]
\[
\alpha = 2\lambda q.
\]
(21)

Equation (21b) is in agreement with Baxter's result, Eq. (3). Similarly, scaling theory implies that
\[
\langle \sigma \rangle \sim \epsilon^{\nu x_\ast} = (\epsilon)^{x_\ast}
\]
on the coexistence curve. Thus, $\beta = \nu x_\ast$ yields
\[
\beta = \frac{1}{2} (1 - q\lambda)
\]
to first order in $\lambda$. Thus, we find all the critical indices by using the assumption that scaling holds.

To check this assumption we use first-order perturbation theory. There is a term $\lambda \sum_r \delta \hat{S}_r \delta \hat{S}_r$ in $\hat{H}$. This term may be written as
\[
\lambda \langle \delta \hat{S}_r^{(1)} \delta \hat{S}_r^{(2)} \rangle + \Delta \delta \hat{S}_r^{(1)} \Delta \delta \hat{S}_r^{(2)}.
\]
(22)

For simplicity, set $K^+ - K^- = K$. Now consider any expectation value $\langle O \rangle_{K,\lambda}$ where $O$ is a product of terms $O^{(1)} O^{(2)}$ with $O^{(1)}$ containing spins on the first sublattice and $O^{(2)}$ containing spins on the second sublattice. To zeroth order, we may write
\[
\langle O \rangle_{K,\lambda} = \langle O^{(1)} \rangle \langle \delta \hat{S}^{(2)} \rangle (\epsilon^0) + \langle O^{(2)} \rangle (\epsilon^0) + O(\lambda).
\]
(23)

Here $\epsilon^0 = K_\ast - K$, with $K_\ast$ being the critical value of $K$ at $\lambda = 0$.

In first-order perturbation theory,
\[
\frac{d}{d\lambda} \langle O \rangle_{K,\lambda} = \langle \delta \hat{S}^{(1)} \delta \hat{S}^{(2)} \rangle (\epsilon^0) \sum_r \langle \Delta \delta \hat{S}_r^{(1)} O^{(1)} \rangle_{K,\lambda=0}.
\]
\[ + \langle \Delta S_{\tau}^{(1)} \rangle \langle O^{(1)} \rangle \langle \Delta S_{\tau}^{(2)} \rangle K_{\lambda \lambda} + \sum_{r} \langle O^{(1)} \Delta S_{\tau}^{(1)} \rangle K_{\lambda \lambda} \langle O^{(1)} \Delta S_{\tau}^{(1)} \rangle K_{\lambda \lambda} = 0. \]  

Since \( \Delta \) is conjugate to \( K \), the first two terms in (24) generate derivatives with respect to \( K \) of \( \langle O^{(1)} \rangle K_{\lambda \lambda} \) and \( \langle O^{(2)} \rangle K_{\lambda \lambda} \). To first order in \( \lambda \), we may replace Eq. (24) by

\[ \langle O^{(1)} \rangle K_{\lambda \lambda} \approx \langle O^{(1)} \rangle \langle \Delta S_{\tau}^{(1)} \rangle \langle O^{(2)} \rangle \langle \Delta S_{\tau}^{(2)} \rangle \right) \right), \]

with

\[ \epsilon^* = \epsilon^0 - \lambda \langle \Delta S \rangle \lambda \lambda. \]  

Equation (28) gives a renormalized \( T - T_c \). Near \( T_c \), \( \langle S \rangle \) is given by

\[ \langle S \rangle = \frac{1}{2} \sqrt{2} e^0 \frac{1}{e^0} + q e^0 \ln |e^0|, \]  

with \( p \) being a new constant and \( q \) being the same as the constant defined by Eq. (16). Eq. (27) can be obtained, e.g., from Eq. (97) of Ref. 8. Note that \( 2K_c(0) = \ln \tan g(\pi/2). \] To first order in \( \lambda \), we can write

\[ \epsilon^* = (e^0 - \lambda \langle \Delta S \rangle \lambda \lambda(1 - q \lambda \ln |e^0|) = \epsilon \left(1 - q \lambda \ln |e^0| \right) = \epsilon \left(1 - q \lambda \right), \]  

with

\[ \epsilon = e^0 - \lambda \langle \Delta S \rangle \lambda \lambda + p \lambda e^0 \]

being essentially \( K_c(\lambda) - K \). The shift in \( K_c \) given by Eq. (29) checks directly against the value given by Baxter's solution.

Equations (25) and (28) may now be used to evaluate critical indices directly. When \( O = O^{(1)} \) and \( T < T_c \), we find, to first order in \( \lambda \),

\[ \langle O^{(1)} \rangle K_{\lambda \lambda} = \pm (\epsilon^* - \epsilon)^\beta_\delta / (\epsilon^* - \epsilon) \]

where \( \beta_\delta \) is the magnetization index for the Onsager solution. This direct solution then recovers the scaling result (21c). Similarly, the two-spin correlation functions which have the form

\[ \langle O^{(2)} \rangle K_{\lambda \lambda} = 1 / \sqrt{1 + \epsilon} \]

when \( \xi^\alpha \sim \xi^\alpha \) become, to first order,

\[ \langle O^{(2)} \rangle K_{\lambda \lambda} = 1 / \sqrt{1 + \epsilon} \sim \epsilon \]

when \( \xi \sim |\epsilon|^{-\nu} \). It follows that the two-spin correlation function has a scaling form to first order in \( \lambda \), and that the coherence length index is correctly given by Eq. (21a). Also, an integration of Eq. (30) over all \( r \) gives

\[ \gamma = 1 + q \lambda, \]

as would be predicted by scaling.
$K_1$ for the two-spin interactions in the first sublattice and an interaction constant $K_2$ in the second sublattice. In first order in $\lambda$ we obtain similarly to Eq. (25)

$$\langle O^{(1)} O^{(2)} \rangle_{K_1 K_2} \approx \langle O^{(1)} \rangle \langle O^{(2)} \rangle \langle \epsilon^*_1 \rangle \langle \epsilon^*_2 \rangle + \lambda \sum_r \langle O^{(1)} \Delta \epsilon^{(1)}_r \rangle \langle O^{(2)} \Delta \delta^{(2)}_r \rangle,$$

with

$$\epsilon^*_1 = \epsilon^*_2 - \lambda \langle \delta_S \rangle_{K_1 K_2}.$$

In first order in $\lambda$ we can write

$$\epsilon^*_1 = \epsilon_1 - \lambda \rho \langle \delta_S \rangle_{K_1 K_2},$$

$$\epsilon^*_2 = \epsilon_2 - \lambda \rho \langle \delta_S \rangle_{K_1 K_2},$$

and similar equations hold for $\epsilon_2$ and $\epsilon^*_2$. Therefore the critical line $\epsilon_1 = 0$ for $\lambda = 0$ moves to $\epsilon^*_1 = 0$ at $\lambda$, that is,

$$\epsilon_1 = \lambda \rho \langle \delta_S \rangle_{K_1 K_2}$$

in first-order perturbation theory. This equation as well as

$$\epsilon_2 = \lambda \rho \langle \delta_S \rangle_{K_1 K_2}$$

can be written in first order in $\lambda$,

$$|\epsilon_1 + \epsilon_2| = |\epsilon_1 - \epsilon_2|^{1 + \lambda \rho \langle \delta_S \rangle_{K_1 K_2}}.$$

Since $\epsilon_1 + \epsilon_2$ and $\epsilon_1 - \epsilon_2$ are conjugate to the energy densities $\delta_x$, the exponent in Eq. (43) is expected to be $(2-x_x)/(2-x_x)$, in agreement with Eq. (20a).

**APPENDIX**

When shown the results of this paper, Wilson drew our attention to a similar problem in field theory which was studied by Wilson, Callen, and Symanzik. In field theory the operator $\Phi^4$ corresponds to the operator $\delta_x$. In the free-field limit $\Phi^4$ has the critical index (dimension) $d$, but in first-order perturbation theory its critical index changes. This leads to a breakdown of scaling.

According to Baxter's solution, the critical index $\alpha$ changes continuously with $\lambda$, Eq. (3). Therefore we expect $\delta_x$ to scale like $1/r^\alpha$ for any $\lambda$. Wilson and Fisher urged us to show this in first-order perturbation theory.

To see this, note that for $r_1 \neq r$ the operator

$$O_\alpha(r_1) \delta_x(r) = \delta_x(r_1) \delta_x(r)$$

is even under the Kramers-Wannier (KW) transformation of one sublattice only. (Under the KW transformation of sublattice 1 $\delta^{(1)}_r$ is odd and $\delta^{(2)}_r$ is even.) Since $\delta_x(r)$ is odd under this transformation, $\alpha$ vanishes. Thus, according to Eq. (13), the critical exponent $\lambda \rho \delta_S$ does not change in first order in $\lambda$.

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1On leave from the Institut für Festkörperforschung of the Kernforschungsanlage Jülich, Germany.


1The relation between the eight-vertex model and the Ising model was independently given by R. Y. Wu, Phys. Rev. B (to be published).

1For positive interaction constants $K^4$ and small $\lambda$ the phase transition takes place for $\alpha = b + c + d$. To obtain $\epsilon \sim (b + c + d - a)/a$ and Eq. (3) we use the symmetry property $Z(\alpha, b, c, d) = Z(c, a, b, d)$, Eq. (11) of C. Fan and F. Y. Wu, Phys. Rev. B 2, 723 (1970).

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