The Electron-Phonon Interaction in Normal and Superconducting Metals\textsuperscript{1}

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I. Correlation Function Approach

This series of lectures has two purposes:
(a) To report on recent developments in the theory of the electron-phonon interaction in metals.
(b) To serve as an introduction to the use of Green's function methods in many particle physics.\textsuperscript{2}

We begin by noticing that one of the very simplest things you can do to a many-particle system is add a particle to it or pull one out. For this reason, it is very convenient to describe the properties of the system in terms of the creation and annihilation operators:

\[ C_{\mu}^\dagger(t') = \text{creation operator} \]
\[ C_{\mu}(t) = \text{annihilation operator}. \]

When these operators act to the right on a state of the system, they respectively add a particle with momentum \( \mu \) to this state at the time \( t' \) and pull one out at the time \( t \).

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\textsuperscript{2} The Green's function methods that I shall employ can be traced back to the work of Martin and Schwoinger (1). These methods are discussed in detail in a book by Baym and Kadanoff (2).

We shall also make use of the wave field operators \( \psi^*(r', t') \) and \( \psi(r, t) \) which respectively add and subtract a particle at the space-time points \( r', t' \) and \( r, t \). If one quantizes in a box of unit volume with periodic boundary conditions the two sets of operators are related by

\[
\begin{align*}
\psi^*(r', t') &= \sum_p \exp(-ip \cdot r') C_p^*(t') \\
\psi(r, t) &= \sum_p \exp(ip \cdot r) C_p(t)
\end{align*}
\] (1.1)

where the momentum sums run over all the allowed momenta in the box. (We use units in which \( \hbar = 1 \).

These are all Heisenberg representation operators and, as such, they have the time dependence

\[
A(t) = \exp(iH't) \quad A(0) = \exp(-iH't)
\] (1.2)

Of course, we may choose the zero point of energy at our convenience. We choose our zero point by using in (1.2)

\[
H' = H - \mu N
\] (1.3)

where \( H \) is the standard Hamiltonian, \( \mu \) the chemical potential, and \( N \) the number operator to be the basic operator which gives the time dependence of our operators.

The two basic correlation functions which we shall discuss are

\[
\begin{align*}
G^+(r, t; r', t') &= \langle \psi(r, t) \psi^*(r', t') \rangle \\
G^-(r, t; r', t') &= \langle \psi^*(r', t') \psi(r, t) \rangle
\end{align*}
\] (1.4)

where \( \langle \rangle \) stands for both a quantum-mechanical average and a statistical average according to the averaging procedure defined by the grand canonical ensemble of statistical mechanics.

Because of the equilibrium nature of the system all physical quantities are independent of the time. Furthermore, our periodic boundary condition in space guarantees the spatial homogeneity of the system. Therefore, \( G^+ \) and \( G^- \) cannot depend upon \( r \) and \( r' \) or \( t \) and \( t' \) individually but only upon the difference variables \( r - r' \), \( t - t' \). This fact
is conveniently included by using the Fourier representations of these quantities as

\[ G^> (r, t; r', t') = G^> (r - r'; t - t') \]

\[ = \sum_p \int \frac{d\omega}{2\pi} \exp \left[ ip \cdot (r - r') - i\omega(t - t') \right] G^> (p, \omega) \]

\[ = \sum_p \exp \left[ ip \cdot (r - r') \right] \langle C_p(t) C^+_p(t') \rangle \]

while

\[ G^< (r - r'; t - t') = \sum_p \exp \left[ ip \cdot (r - r') \right] \langle C^+_p(t') C_p(t) \rangle \]

\[ \sum_p = \int \frac{d\omega}{2\pi} \exp \left[ ip \cdot (r - r') - i\omega(t - t') \right] G^< (p, \omega). \]

\[ G^> (p, \omega) \text{ and } G^< (p, \omega) \text{ have a very direct physical interpretation.} \]

Consider \( G^> (p, \omega) \). We know that \( C_p(t) \) is an operator which removes a particle with momentum at the time \( t \). In the usual quantum-mechanical way, the time Fourier transform of \( C_p(t) \) removes a particle of momentum \( p \) and energy \( \omega \). But, if you are going to remove such a particle, there must be a particle with this energy and momentum present initially. For this reason,

\[ G^> (p, \omega) = \text{density of particles with momentum } p \text{ and energy } \omega. \]  

(1.6)

Because the time Fourier transform of \( C^+_p(t') \) adds a particle with momentum \( p \) and energy \( \omega \), we can see that \( G^> (p, \omega) \) measures the system's ability to accept a particle with this energy and momentum. That is,

\[ G^> (p, \omega) = \text{effective density of states in } p, \omega \]  

(1.7)

or, if you prefer, this may be termed a density of holes.

These statements (1.6) and (1.7) will be very important for our future discussions. Therefore, we examine them in a little detail. From (1.5b), the total number of particles with momentum \( p \), \( N_p = C^+_p C_p \), is given by

\[ \langle N_p \rangle = \langle C^+_p C_p \rangle = \int \frac{d\omega}{2\pi} G^< (p, \omega) \]

(1.8)
which is the integral over all energies of the density as function of energy. Also the total effective density of states is, from (1.5b)

\[ \int \frac{d\omega}{2\pi} G^\ast (p,\omega) = \langle C_p C_p^\ast \rangle . \]

However, the creation and annihilation operators satisfy the equal time commutation or anticommutation relation

\[ C_p(t) C_p^\ast (t) = C_p^\ast (t) C_p(t) = 1 \quad (1.9) \]

where the upper sign here, and in what follows, is appropriate when the particle obey Bose statistics and the lower sign is for Fermi statistics. Therefore, the total density of states is

\[ \int \frac{d\omega}{2\pi} G^\ast (p,\omega) = 1 \pm \langle N^\rho \rangle . \quad (1.10) \]

For fermions as \( \langle N^\rho \rangle \) increases, the effective density of states decreases while for bosons, which are very gregarious, the larger the occupation of a state, the larger is the effective density of states.

To illustrate the meaning of \( G^\ast \) and \( G^\ast \), let us consider a tunneling experiment. In its simplest form this is an experiment in which two conductors are separated by a thin insulating layer and a DC voltage is applied across the gap. The induced current can then be measured as a function of voltage.

Electrons cross the gap via a quantum-mechanical tunneling process. Following many recent authors (3; 4) (see the chapter by R. Prange in
this volume) we can describe the tunneling by a tunneling amplitude
\[ T(p', p) \], which is the probability amplitude for an electron with mo-
momentum \( p \) on one side of the gap reappearing with momentum \( p' \) on
the other. (This is essentially the overlap between the wave functions
on opposite sides of the insulator.)

We calculate the tunneling rate from A to B by making use of the
golden rule, which states that a transition rate is proportional to a matrix
element squared [here, \( |T(p, p')|^2 \)], an energy conserving \( \delta \)-function,
a density of initial and final states, summed over all initial and all final
states. Thus,
\[ \frac{dN}{dt} \bigg|_{A \rightarrow B} = \sum_{pp'} \frac{\hbar}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\omega}{2\pi} |T(p, p')|^2 \]
\[ 2\pi \delta(\omega - \omega' + eV) \ G_A^<(p, \omega) \ G_B^<(p', \omega'). \]  

(1.11a)

Here the \( \delta \)-function requires that the energy change on traversing the
barrier be \( eV \), \( G_A^< \) and \( G_B^< \), of course, represent the density of par-
ticles in A and the density of states in B.

The rate of tunneling in the opposite sense is given by a result identical
to (1.11a) except for the appearance of the density of particles in B and
the density of states in A, that is
\[ \frac{dN}{dt} \bigg|_{B \rightarrow A} = \sum_{pp'} \frac{\hbar}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\omega}{2\pi} |T(p, p')|^2 \]
\[ 2\pi \delta(\omega - \omega' + eV) \ G_A^>(p, \omega) \ G_B^>(p', \omega'). \]  

(1.11b)

To obtain the net tunneling rate we subtract (1.11b) from (1.11a)
to find
\[ \frac{dN}{dt} = \sum_{pp'} \frac{\hbar}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\omega}{2\pi} |T(p, p')|^2 \]
\[ 2\pi \delta(\omega - \omega' + eV) \left[ G_A^<(p, \omega) \ G_B^<(p', \omega') - G_A^>(p, \omega) \ G_B^>(p', \omega) \right]. \]  

(1.12)

Equation (1.12) may be used to predict the tunneling rate in all cases
except that in which both conductors are in a superconducting state.
In that case, some very exceptional behavior results as first predicted by
Josephson (5) and experimentally verified by Anderson and Rowell (6).
This kind of tunneling is discussed in detail by P. W. Anderson in this volume.

In all but this exceptional case, one expects that, as $V \to 0$, the tunneling current should vanish. However, if $A$ and $B$ are different materials so that $G_A^\varsigma \neq G_B^\varsigma$, Eq. (1.12) does not appear to predict the vanishing of the current at $V \to 0$.

In order to see how this vanishing does in fact occur, we must make use of a kind of detailed balancing condition which relates the density of particles and the density of states. It is a characteristic feature of a system in equilibrium at temperature $T$ that the relative occupation of a state with energy $\omega$ is proportional to $\exp(-\beta \omega)$, where $\beta = (kT)^{-1}$, and $k$ is the Boltzmann constant. This characteristic feature shows up as a restriction upon $G^\varsigma$ and $G^\varrho$ that, for a system in equilibrium,

$$G^\varsigma (p, \omega) = \exp(-\beta \omega) \, G^\varrho (p, \omega)$$

(density of particles) = $\exp(-\beta \omega)$ (density of states).

We shall prove this condition in a moment. For now let us see its consequences in the tunneling rate given by (1.12). We use (1.13) to eliminate $G_A^\varsigma$ in favor of $G_A^\varrho$ and $G_B^\varsigma$ in favor of $G_B^\varrho$. Then (1.12) becomes

$$\frac{dN}{dt} = \sum \int \int |T|^2 \, 2\pi \delta(\omega - \omega') \, e \left[ \exp(-\beta \omega) - \exp(-\beta \omega') \right].$$

Clearly now as $V \to 0$, so does the current.

Equation (1.13) will be quite crucial for us so we consider its proof in some detail. In the ground canonical ensemble, the expectation value of any operator is given by

$$\langle X \rangle = \sum \exp \left[ -\beta(E - \mu N) \right] \langle \xi | X | \xi \rangle / Z_G$$

$$= \text{Tr} \left[ \exp(-\beta H') X \right] / Z_G$$

(1.14)

where the sum runs over all possible states with any number of particles and where

$$Z_G = \sum \exp \left[ -\beta(E - \mu N) \right] = \text{Tr} \left[ \exp(-\beta H') \right]$$

is the grand partition function.
Now consider the two functions

\[ F^x(t) = \langle A(t) X \rangle \]
\[ F^z(t) = \langle X A(t) \rangle \]  \hspace{1cm} (1.15)

If \( A(t) \) is a Heisenberg representation operator with no explicit time dependence

\[ A(t) = \exp(iH't\, A(0) \exp(-iH't) \]  \hspace{1cm} (1.2)

Equation (1.2) can be used to define \( A(t) \) for complex values of the time. Now consider, in particular,

\[ F^x(t) \big|_{t=-i\omega} = \langle \exp(-\beta H') A(0) \exp(-\beta H') X \rangle\]
\[ = \text{Tr} \left[ \exp(-\beta H') \exp(\beta H') A(0) \exp(-\beta H') X \right] Z_0 \]
\[ = \text{Tr} \left[ A(0) \exp(-\beta H') X \right] Z_0 \]
\[ = \text{Tr} \left[ \exp(-\beta H') X A(0) \right] Z_0 = \langle X A(0) \rangle . \]

Consequently,

\[ F^x(t) \big|_{t=-i\omega} = F^z(t) \big|_{t-\omega} \]  \hspace{1cm} (1.16)

Let us apply this theorem to \( G^x \) and \( G^z \). Equation (1.16) implies that

\[ G^x(t, t'; r, r') \big|_{t=-i\omega} = G^z(r, t; r', t') \big|_{t+\omega} \]
\[ \int \frac{d\omega}{2\pi} \exp[-i\omega(t-t')] G^z(p, \omega) \big|_{t-i\omega} \]  \hspace{1cm} (1.17)
\[ = \int \frac{d\omega}{2\pi} \exp[-i\omega(t-t')] G^z(p, \omega) \big|_{t+\omega} \]

or

\[ \int \frac{d\omega}{2\pi} \exp(i\omega t') \exp(-\beta \omega) G^z(p, \omega) = \int \frac{d\omega}{2\pi} \exp(i\omega t') G^z(p, \omega), \]

which immediately leads us back to (1.13).

The functions \( G^x(p, \omega) \) and \( G^z(p, \omega) \) contain a tremendous amount of useful information. They describe, as we have seen, the density of
particles and the density of states, and they describe all the thermodynamic properties of the system. Consequently, it is quite worthwhile to know these functions.

In actual calculations, it is very convenient to make use of the spectral weight function

\[ A(p, \omega) = G^>(p, \omega) \mp G^<(p, \omega) \]  

(1.18)

= Fourier transform \( \langle C^\dagger_p(t)C_p(t') \mp C^\dagger_p(t'C_p(t) \rangle \).

The usefulness of \( A(p, \omega) \) is derived from the equal time commutation relation (1.7), which implies the sum rule

\[ \int \frac{d\omega}{2\pi} A(p, \omega) = 1 . \]  

(1.19)

By making use of the detailed balancing condition (1.13) together with the definition (1.17) we can express \( G^> \) and \( G^< \) in terms of \( A \) as

\[ G^>(p, \omega) = [1 \pm f(\omega)] A(p, \omega) \]  

(1.20)

\[ G^<(p, \omega) = f(\omega) A(p, \omega) \]

where

\[ f(\omega) = \frac{1}{\exp(\beta \omega) + 1} \]  

(1.21)

is the a priori probability of observing a particle with energy \( \omega \) in the grand canonical ensemble.

Now we must face the hard job of determining \( G^> \) and \( G^< \). To see how this goes consider just the trivial case of free particles. In general, \( \omega \) represents the total energy of a particle added to the system, its kinetic energy plus the interaction energy that it aims through its interaction with all the other particles in the system. However, for noninteracting particles \( \omega \) must be just the kinetic energy

\[ \epsilon_p = p^2/2m - \mu . \]  

(1.22)

Therefore \( G^> \), \( G^< \), and \( A \) are all proportional to \( \delta(\omega - \epsilon_p) \). However, for \( A(p, \omega) \), we have a sum rule (1.18) which tells us the constant of proportionality, so that

\[ A(p, \omega) = 2\pi \delta(\omega - \epsilon_p) . \]  

(1.23)
Using (1.22), we can easily calculate the density of particles with momentum \( p \) as

\[
\langle N_p \rangle = \int \frac{d\omega}{2\pi} \ G^<(p, \omega) = \int \frac{d\omega}{2\pi} \ f(\omega) \ A(p, \omega)
\]

\[
= f(\varepsilon_p)
\]

\[
= \frac{1}{\exp [\beta(p^2/2m - \mu)] + 1}
\]

the familiar result for a system of noninteracting fermions or bosons.

In generalizing these free particle results, it is convenient for us to define a function of a complex variable \( z \):

\[
G(p, z) = \int \frac{d\omega}{2\pi} \ \frac{A(p, \omega)}{z - \omega}.
\]  \hspace{1cm} (1.24)

For the free-particle system:

\[
G(p, z) = \frac{1}{z - \varepsilon_p}.
\]  \hspace{1cm} (1.25)

In the general case,

\[
G(p, z) = \frac{1}{z - \varepsilon_p - \Sigma(p, z)}.
\]  \hspace{1cm} (1.26)

Since the variable \( \omega \) represents the total energy of an added particle we can interpret \( \Sigma(p, \omega) \) [that is, \( \Sigma(p, z) \) for real \( z \)] as the extra interaction energy that would be produced by the addition of a hypothetical particle of momentum \( p \) and energy \( \omega \). For this reason \( \Sigma \) is termed the self-energy. If the hypothetical energy is to be a realizable energy difference \( \omega \) must satisfy the dispersion relation

\[
\omega = \varepsilon_p + \Sigma(p, \omega).
\]  \hspace{1cm} (1.27)

Notice that these energy levels give the position of the poles of \( G(p, z) \) and hence of \( \delta \)-function singularities in \( A(p, \omega) \).

To illustrate how \( \Sigma(p, \omega) \) may be calculated, let us consider the interaction of electrons with longitudinal phonons. In the simplest model,
this interaction may be represented by the interaction Hamiltonian

\[ H_{cp} = \sum_q \left[ v_q a_q^+ \int dr \exp (i \mathbf{q} \cdot \mathbf{r}) \psi^*(r) \psi(r) + v_q^* a_q \int dr \exp (-i \mathbf{q} \cdot \mathbf{r}) \psi^*(r) \psi(r) \right]. \] (1.28)

Here \( a_q^+ \) and \( a_q \) are phonon creation and annihilation operators, and \( v_q \) is a matrix element which measures the coupling strength. We can rewrite (1.27) as

\[ H_{cp} = \sum_{pq} (v_q a_q C_{p\rightarrow q}^+ C_p + v_q^* a_q^+ C_{p\rightarrow q} C_p) \] (1.29)

The first term in (1.29) describes a process in which an electron with momentum \( p \) absorbs a phonon with momentum \( q \) and hence scatters into the momentum state \( p + q \). The second term describes a similar process in which a phonon is emitted.

As a preliminary to the calculation of \( \Sigma(p, \omega) \), we calculate \( \langle H_{cp} \rangle \) in second order perturbation theory. As usual, the second order perturbation theory gives a sum over all initial and the final states of a matrix element squared divided by an energy denominator which is the difference in energy between the initial and the final state. In this case

\[ \langle H_{cp} \rangle = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \sum_{p, p'} G^e(p, \omega) G^c(p', \omega') \left| v_q \right|^2 \]

\[ \left[ \frac{N_q}{\omega + \omega_q - \omega'} + \frac{N_q + 1}{\omega - \omega_q - \omega'} \right]_{\omega = \omega_q, p = P_p} \] (1.30)

Here \( \omega_q \) is the energy of a phonon with wave vector \( q \), and

\[ N_q = \frac{1}{\exp(\beta \omega_q) - 1} \]

is the equilibrium number of phonons with wave vector \( q \). The first term in the square bracket of (1.30) describes a process in which an electron with \( (p, \omega) \) absorbs a phonon and scatters into the state \( (p', \omega') \). The energy difference between the initial state and the final state is \( \omega + \omega_q - \omega' \). The density of initial states is given by the density of
electrons, $G^<(p', \omega')$ times the density of phonons, $N_q$, and the density of final states if $G^>(p', \omega')$. In just the same way, the other term in the square bracket gives the energy shift due to phonon emission processes.

![Fig. 2](image)

We calculate the self-energy as the change in the interaction energy which occurs upon the addition to the system of a particle with momentum $p$ and energy $\omega$. This addition may be represented by changing the density of particles according to

$$G^<(p', \omega') \rightarrow G^<(p', \omega') + 2\pi\delta(\omega - \omega') \delta_{p, p'}.$$  \hspace{1cm} (1.31a)

The added particle also changes the density of states. Since an electron is a fermion, the change is a reduction in the density of states according to

$$G^>(p', \omega') \rightarrow G^>(p', \omega') - 2\pi\delta(\omega - \omega') \delta_{p, p'}.$$  \hspace{1cm} (1.31b)

The self-energy is then defined as

$$\Sigma(p, \omega) = \delta \langle H_{\text{ep}} \rangle$$  \hspace{1cm} (1.32)

$$= \int \frac{d\omega'}{2\pi} \sum_{p'} |v_q|^2 \left\{ G^>(p', \omega') \left[ \frac{N_q}{\omega + \omega_q - \omega'} + \frac{N_q + 1}{\omega - \omega_q - \omega'} \right] + G^<(p', \omega') \left[ \frac{N_q}{\omega - \omega_q - \omega'} + \frac{N_q + 1}{\omega + \omega_q - \omega'} \right] \right\}_{\pm=1-\pm_p'}.$$  

This result can again be understood in terms of second order perturbation theory.3

3 The argument given here may be extended, but only with considerable care, to obtain $\Sigma$ as a variational derivative of an object related to $\langle H_{\text{ep}} \rangle$ with respect to $G$. A rigorous version of this argument is presented by Baym (7).
Before we go any further, we must face a serious difficulty. $\Sigma(p, \omega)$ is, strictly speaking, undefined at a fantastic number of different points on the real axis: all those points at which the energy denominator may vanish. The exact position of these singularities depends in great detail upon the exact size and shape of the system. We seek a method of eliminating this unwanted detail and retaining only that part of the information contained in (1.32) which pertains to the properties of a very large system. 

There is a very simple method for accomplishing this elimination. We simply replace the $\omega$ which appears in (1.32) by the complex variable $z$. Then (1.32) becomes

$$
\Sigma(p, z) = \int \frac{d\omega'}{2\pi} \sum_{p'} |v_{q'}|^2 \left\{ G^c(p', \omega') \left[ \frac{N_q}{z + \omega_q - \omega'} + \frac{N_q + 1}{z - \omega_q - \omega'} \right] + G^c(p', \omega') \left[ \frac{N_q}{z - \omega_q - \omega'} + \frac{N_q + 1}{z + \omega_q - \omega'} \right] \right\}_{q' = |p - p'|}.
$$

By taking $z$ to have an imaginary part we stand far enough away from the singularities so that we may consider them to be essentially continuously distributed along the real axis. In fact, we may allow them to become continuously distributed by taking the limit as the volume of the system goes to infinity and

$$
\sum_{p'} \rightarrow \int \frac{d^3p'}{(2\pi)^3}.
$$

Then, $\Sigma(p, z)$ has the form:

$$
\Sigma(p, z) = \int \frac{d\omega}{2\pi} \frac{\Gamma(p, \omega)}{z - \omega}.
$$

with $\Gamma(p, \omega)$ now being a continuous function of $\omega$. 

II. Green's Function Approach

At this point, we shall consider how the results we have just obtained may be rederived by using a more formal Green's function technique. To do this, we define

\[ G(r, t; r', t') = \frac{1}{i} \langle T [\psi(r, t) \psi^*(r', t')] \rangle \]  

(2.1)

where the time variables are limited to be on the imaginary axis with

\[ 0 < it < \beta ; \quad 0 < it' < \beta . \]

Here \( T \) is a Wick time ordering symbol which tells you to order the operators according to the sense of the "time" \( it \): the operator with the larger value of \( it \) appearing on the left. Also, for fermions, \( t \) contains a factor of \((-i)\) for each permutation of the operators from their standard order. Thus,

\[ G(r, t; r', t') = \begin{cases} 
\frac{1}{i} G^>(r, t; r', t') & \text{for } it > it' \\
\pm \frac{1}{i} G^<(r, t; r', t') & \text{for } it < it' .
\end{cases} \]  

(2.2)

The main reason for defining \( G \) for imaginary times lies in the fact that we can relate the values of \( G \) at the two end points of its region of definition by using Eq. (1.16). This implies

\[ G(r, t; r', t') \bigg|_{t=0} = \pm G(r, t; r', t') \bigg|_{t=-i\beta} . \]  

(2.3)

The boundary condition (2.3) is most conveniently represented by writing \( G \) as a Fourier series in its time variable as

\[ G(r, t; r', t') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{i\beta} \sum_2 \exp \left[ ip \cdot (r - r') - iz_c(t - t') \right] G_k(p) . \]  

(2.4)

This form for \( G \) will necessarily satisfy the boundary condition if

\[ z_c = \frac{\pi v}{-i\beta} \]

\[ r = \{ \text{even integer for bosons} \} \]

\[ \{ \text{odd integer for fermions} . \]
By inverting the Fourier series, one can easily show that the Fourier coefficient is

$$G_n(p) = G(p, z_n) = \int \frac{d\omega}{2\pi} \frac{A(p, \omega)}{z_n - \omega}.$$ (2.5)

A useful technique for evaluating $G$ involves the equation of motion of the annihilation operator:

$$i \frac{\partial \psi(r, t)}{\partial t} = -[H', \psi(r, t)].$$

For free particles this equation of motion is

$$\left[i \frac{\partial}{\partial t} + \frac{p^2}{2m} + \mu\right] \psi(r, t) = 0.$$

In calculating the time derivative of $G(r, t; r', t')$, we use Eq. (2.4) together with the time derivative of the discontinuity at $it = it'$ produced by the time ordering. This discontinuity is

$$\frac{1}{i} \langle \psi(r, t) \psi^*(r', t) \mp \psi^*(r', t) \psi(r, t) \rangle = \frac{1}{i} \delta(r - r').$$

Thus for free particles $G$ obeys:

$$\left[i \frac{\partial}{\partial t} + \frac{2}{2m} + \mu\right] G(r, t; r', t') = \delta(r - r') \delta(it - it').$$ (2.6)

To find $G(p, z)$ from (2.6), we multiply by $\exp \left[-ip \cdot (r - r') + iz(t - t')\right]$ and integrate over all space and all time in the interval $[0, -i\beta]$. Then, we find

$$[z_n - \epsilon_n] G(p, z_n) = 1$$

or

$$G(p, z_n) = \frac{1}{z_n - \epsilon_n} = \int \frac{d\omega}{2\pi} \frac{A(p, \omega)}{z_n - \omega}.\]$$

There exists a theorem which states that if an equation that is $F(z)$ and $G(z)$ are two analytic functions of $z$ such that $F(z) = G(z)$ holds for all...
\[ z_q = (\pi r - i\beta) \text{ for even or odd integral } r \text{ and if } F(z) \text{ and } G(z) \text{ have no essential singularity at infinity then} \]

\[ F(z) = G(z) \]

for all \( z \). Therefore (2.6) implies

\[ \int \frac{d\omega}{2\pi} \frac{A(p, \omega) }{z - \omega} = \frac{1}{z - \epsilon_p} \]

for all \( z \) and we find

\[ A(p, \omega) = 2\pi \delta(\omega - \epsilon_p) \]

just as before.

To illustrate the application of Green's function techniques, we consider the interaction of electrons with longitudinal phonon, which may be represented by the Hamiltonian

\[ H' = H_e + H_p + H_{ep} \]

\[ H_e = \int dr \overline{\psi^*}(r) \left[ \frac{V^2}{2m} - \mu \right] \psi(r) \]  \hspace{1cm} (2.7a)

\[ H_p = \sum_q \omega_q \overline{a_q} a_q \]  \hspace{1cm} (2.7b)

\[ H_{ep} = \sum_q \nu_q \overline{a_q} \int dr \exp{( - i q \cdot r )} \psi^*(r) \psi(r) \]

\[ + \sum_q \nu_q^* \overline{a_q^*} \int dr \exp{( + i q \cdot r )} \psi^*(r) \psi(r) \]  \hspace{1cm} (2.7d)

Here \( \psi^*(r) \) and \( \psi(r) \) are electron wave field creation and annihilation operators, \( a_q^* \) and \( a_q \) are phonon creation and annihilation operators, \( \omega_q \) is the phonon energy. For simplicity, we ignore Umklapp processes, the possibility of several types of phonons, and the spin of the electrons.

We see immediately that the various operators obey the equations of motion

\[ \left[ i \frac{\partial}{\partial t} + \frac{V^2}{2m} + \mu \right] \psi(r, t) \]  \hspace{1cm} (2.8a)

\[ = \sum_q \left[ \nu_q \overline{a_q}(t) \exp(iq \cdot r) + \nu_q^* \overline{a_q^*}(t) \exp(-iq \cdot r) \right] \psi(r, t) \]
\[ (i \frac{\partial}{\partial t} - \omega_q) a_q(t) = \int dr \nu_q^* \exp (i q \cdot r) \psi^*(r, t) \psi(r, t) \] \tag{2.8b} \\
\[ (i \frac{\partial}{\partial t} + \omega_q) a_q^+(t) = -\int dr \nu_q \exp (-i q \cdot r) \psi^*(t, r) \psi(r, t) . \] \tag{2.8c} 

It is quite simple to compute the equation of motion of the electronic Green's function. We have:

\[
\left[ \frac{i}{\hbar} \frac{\partial}{\partial t} + \frac{\mathbf{p}^2}{2m} + \mu \right] G(r, t; r', t') = \delta(r - r') \delta(t - t') \\
+ \frac{1}{i} \left< T \left\{ \sum_q \nu_q a_q(t) \exp (i q \cdot r) \ + \nu_q^* a_q^+(t) \exp (-i q \cdot r) \right\} \psi(r, t) \psi^*(r', t') \right> . \tag{2.9} 
\]

Therefore, in order to find \( G(t, t') \) we need to know

\[ \left< T[a_q(t) \psi(1) \psi^*(1')] \right> \]

and

\[ \left< T[a_q^+(t) \psi(1) \psi^*(1')] \right> \]

where \( t \) stands for \( r, t \), and \( 1' \) for \( r', t' \). These functions are not known, but we can use the equation of motion for \( a_q(t) \) and \( a_q^+(t') \) to eliminate the phonon variables from these expressions. If we are to make use of these equations of motion, we must allow the time in \( a_q(t) \) and \( a_q^+(t') \) to be different from \( t \), so that we may differentiate with respect to \( t \). For this reason, we define:

\[ F_q(t, 1, 1') = \left< T[a_q(t) \psi(1) \psi^*(1')] \right> \] \tag{2.10a} \\
\[ F_q'(t, 1, 1') = \left< T[a_q^+(t) \psi(1) \psi^*(1')] \right> . \] \tag{2.10b} 

From (2.3b) and (2.3c) we find

\[ \left[ i \frac{\partial}{\partial t} - \omega_q \right] F_q(t, 1, 1') \]

\[ = \nu_q^* \left< T \int d\bar{r} \exp (-i q \cdot \bar{r}) \psi^*(\bar{r}, t) \psi(\bar{r}, t) \psi(t) \psi^*(1') \right> . \] \tag{2.11a}
and
\[
\left[ i \frac{\partial}{\partial t} + \omega_q \right] \vec{F}_q(t, 1, 1') = -v_q \langle T \rangle \int d\vec{r} \exp \left( iq \cdot \vec{r} \right) \psi^*(\vec{r}, 1) \psi^*(\vec{r}, t) \psi^*(1') \psi(1') \right] .
\] (2.11b)

There are no discontinuity terms coming from differentiating $T$ because the equal time commutator of $a_q$ and $a_q^*$ with $\psi$ and $\psi^*$ vanishes.

Equations (2.11) are to be solved with the standard 0 to $-i\beta$ boundary condition:
\[
F_q(t, 1, 1') \big|_{t=0} = F_q(t, 1, 1') \big|_{t=-\beta} \quad (1.16)
\]

and similarly for $F$. In obtaining the solution to (2.11) it is convenient to use the function $d_{\nu}(t, t')$, which is defined by the differential equation
\[
\left[ i \frac{\partial}{\partial t} - \omega_q \right] d_{\nu}(t, t') = \delta(t - t') \quad \text{for} \quad 0 < \nu t < \beta
\]
and the boundary condition:
\[
d_{\nu}(0, t') = d_{\nu}(-i\beta, t').
\]

The solution to this equation for $d_{\nu}$ may be written down at once. We split $d_{\nu}$ into two parts in just the same way as we divided $G(1, 1')$ into two parts by writing
\[
id_{\nu}(t_1, t_2) = \begin{cases} d_{\nu}^>(t_1, t_2) & \text{for} \quad i\nu t_1 > i\nu t_2 \\ d_{\nu}^< (t_1, t_2) & \text{for} \quad i\nu t_1 < i\nu t_2. \end{cases} \quad (2.12)
\]

We can see immediately that $d_{\nu}^>$ and $d_{\nu}^<$ are exactly analogous to $G^>(p, t_1 - t_2)$ and $G^<(p, t_1 - t_2)$ for free bosons if the replacement
\[
\left( \frac{p^2}{2m} - \mu \right) \to \omega
\]
is made. Therefore, we have
\[
d_{\nu}^>(t_1 - t_2) = \exp \left[ -i\nu(t_1 - t_2) \right] (N_q + 1)
\]
\[
d_{\nu}^<(t_1 - t_2) = \exp \left[ -i\nu(t_1 - t_2) \right] N_q \quad (2.13a)
\]
where \( N_q \) is the equilibrium number of phonons with wave vector \( q \).

\[
N_q = \frac{1}{\exp(\beta \omega_q) - 1}.
\]

Also, we see

\[
d_{>\omega_q}(t_1 - t_2) = -\exp\left[\im \omega_q (t_1 - t_2)\right] N_q
\]
\[
d_{<\omega_q}(t_1 - t_2) = -\exp\left[\im \omega_q (t_1 - t_2)\right] (N_q + 1)
\]

since

\[
\frac{1}{\exp(\beta \omega_q) - 1} = -\frac{1}{\exp(\beta \omega_q) - 1} - 1 = -(1 + N_q).
\]

We can see immediately that solution to Eq. (2.11) with the boundary condition (1.16) may be expressed in terms of \( d \) as:

\[
F(t, 1, 1') = \int d\vec{r} \int_0^{-d} d\vec{r}_q \exp(-i \vec{q} \cdot \vec{r}) \langle \psi^*(\vec{r}, \vec{r}_q) \psi(1) \psi(1') \rangle
\]  

\[
\langle T[y^*(\vec{r}, \vec{r}_q) \psi(\vec{r}, \vec{r}_q) \psi(1) \psi(1')] \rangle
\]

\[
\bar{F}(t, 1, 1') = -\int d\vec{r} \int_0^{-d} d\vec{r}_q \exp(i \vec{q} \cdot \vec{r}) \langle \psi^*(\vec{r}, \vec{r}_q) \psi(1) \psi(1') \rangle
\]

\[
\langle T[y^*(\vec{r}, \vec{r}_q) \psi(\vec{r}, \vec{r}_q) \psi(1) \psi(1')] \rangle.
\]

Notice the great utility of the boundary condition and the \([0, -i\beta]\) approach. This approach enables us to eliminate all reference to the phonon variables in expressions like \( F \) and \( \bar{F} \) and reduce these to expressions which involve only electron operators. Of course, we have to pay a price for this elimination. This price may be seen from the result of substituting (2.14) and back into the equation of motion of (2.6):

\[
\left[ -\frac{\partial}{\partial t} + \frac{V^2}{2m} + \mu \right] G(1, 1') = \delta(1 - 1')
\]

\[
+ \frac{1}{i} \sum_q |v_q|^2 \int d\vec{r} \int_0^{-d} d\vec{r}_q \exp(i \vec{q} \cdot (\vec{r} - \vec{r}_q)) \langle \psi^*(\vec{r}, \vec{r}_q) \psi(\vec{r}, \vec{r}_q) \psi(1) \psi(1') \rangle.
\]

\[
\langle T[y^*(\vec{r}, \vec{r}_q) \psi(\vec{r}, \vec{r}_q) \psi(1) \psi(1')] \rangle.
\]
It is convenient to rewrite this expression in terms of

\[ G_d(1, 2; 1', 2') = \left( -\frac{1}{i} \right)^2 \langle T[\psi(1) \psi(2) \psi^*(2') \psi^*(1')] \rangle \]  

(2.16)

as

\[ \left[ i \frac{\partial}{\partial t} + \frac{p^2}{2m} + \mu \right] G(1, 1') = \delta(1 - 1') \]

\[ \pm \int dr_2 \int_0^{-i\beta} dr_1' V(1 - 2) G_d(1, 2; 1', 2) \]  

(2.17)

where

\[ V(1 - 2) = \sum_q |v_q|^2 \left[ \exp \left[ iq \cdot (r_1 - r_2) \right] d_{vq}(t_1, t_2) \right] \]

\[ \exp \left[ -iq \cdot (r_1 - r_2) \right] d_{-vq}(t_1, t_2) \].

(2.18)

The reason that Eq. (2.17) is so instructive is that it is quite similar in structure to the equation of motion which would emerge from an ordinary electron-electron interaction. If the ordinary interaction can be represented by a central potential \( v(|r_1 - r_2|) \), then the equation of motion for \( G \) again takes the form (2.13) except for this case:

\[ V(1, 2) = \delta(t_1 - t_2) v(|r_1 - r_2|). \]  

(2.19)

Therefore the phonons act to produce an affective electron-electron potential. This potential may be considered to result from the fact that an electron at one point in space and time may change the phonon field by emitting or absorbing phonons. This change in the phonon field can at a later time, scatter electrons at another point in space so that the scattering looks like it is produced by a retarded interaction between electrons.

To find \( G \) from Eq. (2.17), we must make some approximation for the \( G_d(1, 2; 1', 2') \) which appears in that equation. This two-particle Green's function, of course, describes the correlated motion of two electrons added to the system at \( 1' \) and \( 2' \) and removed at \( 1 \) and \( 2 \). The simplest possible approximation for this \( G_d \) is to come from the assumption that the added electrons move quite independently of one another. This assumption may be expressed mathematically as

\[ G_d(1, 2; 1', 2') = G(1, 1') \ G(2, 2') - G(1, 2') \ G(2, 1'). \]  

(2.20)
The appearance of two terms in this expression is a result of Fermi statistics obeyed by the electrons which requires that

\[ G_2(1, 2; 1', 2') = - G_2(2, 1; 1', 2') . \]

When we substitute (2.20) in (2.17) we find

\[
\left[ i \frac{\partial}{\partial t} + \frac{p^2}{2m} + \mu \right] G(1, 1') = \delta(1 - 1') \tag{2.21}
\]

\[
\pm i \int V(1 - 2) G(1, 2) \, d2 \, G(1, 1')
\]

\[
+ i \int d2 \, V(1 - 2) \, G(1, 2) \, G(2, 1')
\]

the middle term on the right-hand side of (2.21) should be neglected for two reasons:

(a) This term is proportional to

\[
\int d2 \, V(1 - 2) \, \langle n(2) \rangle = \langle n \rangle \int d2 \, \int_0^{i\beta} dt \, V(r, t).
\]

After the \( r_2 \) integral is performed, this expression becomes proportional to

\[
| \psi |^2 | \psi = 0 .
\]

However, a \( q = 0 \) phonon is pure nonsense. This "phonon" represents the effect of picking up and bodily displacing the whole crystal. Therefore this whole term is physically meaningless.

(b) This term is proportional to the average potential field produced by all the electrons in the system. As such, it has actually been included in the original definition of the band structure which underlies our original Hamiltonian. Therefore, this term has already been counted and it must now be left out.

After this term is thrown out Eq. (2.21) may be rewritten as:

\[
\left[ i \frac{\partial}{\partial t} + \frac{p^2}{2m} + \mu \right] G(1, 1') = \delta(1 - 1') + \int d2 \, \Sigma(1, 2) \, G(2, 1') \tag{2.22}
\]

where the self-energy \( \Sigma \) is

\[
\Sigma(1, 1') = V(1 - 1') \, G(1, 1'). \tag{2.23}
\]
It is a trivial matter to verify that $\Sigma(1, 1')$ obeys the same boundary conditions as $G$, i.e.,

$$\Sigma(1, 1')\big|_{t_i = 0} = - \Sigma(1, 1')\big|_{t_i = \beta}$$

(2.24)

since $\nu(1, 1')$ obeys the boson boundary condition

$$\nu(1, 1')\big|_{t_i = 0} = \nu(1, 1')\big|_{t_i = \alpha - \omega}. $$

Since $\Sigma$ obeys the same boundary condition as $G$, its formal properties are very similar to the formal properties of $G$. For example, when $\Sigma(1, 1')$ is split into two parts as

$$\Sigma^{+}(1 - 1') = \int \frac{d^3p}{(2\pi)^3}\frac{d\omega}{2\pi}$$

$$i \Sigma(1, 1') = \begin{cases} 
\exp [ip \cdot (r_1 - r_1') - i\omega(t_1 - t_1')] \Sigma^{+}(p, \omega) & \text{for } t_1 > t_1', \\
-\Sigma^{+}(1 - 1') - \int \frac{d^3p}{(2\pi)^3}\frac{d\omega}{2\pi} \exp [ip \cdot (r_1 - r_1') - i\omega(t_1 - t_1')] \Sigma^{+}(p, \omega) & \text{for } t_1 < t_1'
\end{cases}$$

then $\Sigma^{+}(p, \omega)$ and $\Sigma^{-}(p, \omega)$ obey the detailed balancing relation

$$\Sigma^{+}(p, \omega) = \exp (-\beta\omega) \Sigma^{-}(p, \omega);$$

also, $\Sigma$ may be written as a Fourier series

$$\Sigma(1, 1') = \int \frac{d^3p}{(2\pi)^3} \exp [ip \cdot (r_1 - r_1')] \frac{1}{-i\beta} \Sigma(p)$$

(2.26)

where the Fourier coefficient given by

$$\Sigma(p) = \Sigma(p, z) = \int \frac{d\omega}{2\pi} \frac{\Sigma^{+}(p, \omega) + \Sigma^{-}(p, \omega)}{z_1 - \omega} = \int \frac{d\omega}{2\pi} \frac{\Gamma(p, \omega)}{z_1 - \omega}. $$

This Fourier series representation of $\Sigma(1, 1')$ is very convenient because it enables us to solve the differential Equation (2.22) quite directly.
If we multiply this equation by $\exp \left[ -i p \cdot (r_1 - r'_1) + i z (t_1 - t'_1) \right]$ and then integrate over all $r_1$ and all $t_1$ in the interval $0 < it_1 < \beta$, then (2.22) becomes

$$[z_e - \varepsilon_p - \Sigma(p, z)] G(p, z) = 1.$$  \hspace{1cm} (2.28)

Equation (2.28) is a relation between the analytic functions

$$\Sigma(p, z) = \int \frac{d\omega}{2\pi} \frac{\Sigma^*(p, \omega) + \Sigma^c(p, \omega)}{z - \omega} = \int \frac{d\omega}{2\pi} \frac{I(p, \omega)}{z - \omega}$$  \hspace{1cm} (2.29a)

$$G(p, z) = \int \frac{d\omega}{2\pi} \frac{G^*(p, \omega) + G^c(p, \omega)}{z - \omega} = \int \frac{d\omega}{2\pi} \frac{A(p, \omega)}{z - \omega}$$  \hspace{1cm} (2.29b)

which is known to hold on all the points

$$z = z_e = \frac{\pi v}{-i \beta}, \quad r = \text{odd integer}.$$  

From the fact that this relation holds on this limited set of points and the fact that neither $\Sigma(p, z)$ nor $G(p, z)$ has an essential singularity at $\infty$ it follows that (2.28) holds for all $z$, i.e., that

$$G(p, z) = \frac{1}{z - \varepsilon_p - \Sigma(p, z)}.$$  \hspace{1cm} (2.30)

In order to make use of Eq. (2.30) we use (2.18) and (2.23) to write

$$\Sigma^c(1, 1') = V^c(1, 1') G^c(1, 1')$$

$$= \sum_q \left| v_q \right|^2 \exp \left[ iq \cdot (r_1 - r'_1) \right] d_{v_q}(t_1 - t'_1)$$

$$- \exp \left[ -i q \cdot (r_1 - r'_1) \right] d_{-v_q}(t_1 - t'_1) G^c(1, 1').$$  \hspace{1cm} (2.31)

The Fourier transform of (2.31) is

$$\Sigma^c(p, \omega) = \int \frac{d^3 p'}{(2\pi)^3} \frac{d\omega'}{2\pi} G^c(p', \omega') \left| v_{q'} \right|^2 \left[ N_q 2\pi \delta(\omega + \omega_q - \omega') \right.$$

$$+ \left( N_q + 1 \right) 2\pi \delta(\omega - \omega_q - \omega') \right] \quad q = |p - p'|.$$  \hspace{1cm} (2.32a)
A similar evaluation gives

$$
\Sigma^r(p, \omega) = \int \frac{dp'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \ G^r(p', \omega') \ |v_q|^2 \left[ ((N_q + 1)2\pi\delta(\omega - \omega_q - \omega')) + N_q 2\pi\delta(\omega - \omega_q - \omega') \right] \quad q = |p - p'|.
$$

(2.32b)

The substitution of (2.32) into (2.29a) leads back to our earlier result (1.33):

$$
\Sigma(p, z) = \int \frac{d\omega'}{2\pi} \int \frac{dp'}{(2\pi)^3} \ |v_q|^2 \left\{ G^r(p', \omega') \left[ \frac{N_q}{z + \omega_q - \omega'} + \frac{N_q + i}{z - \omega_q - \omega'} \right] + G^c(p', \omega') \left[ \frac{N_q}{z - \omega_q - \omega'} + \frac{N_q + 1}{z + \omega_q - \omega'} \right] \right\} \quad q = |p - p'|.
$$

(2.33)

It is now quite simple to derive an expression for \(A(p, \omega)\) since \(iA(p, \omega)\) is the discontinuity in \(G(p, z)\) as \(z\) crosses the real axis. From (2.30),

$$
A(p, \omega) = \frac{\Gamma'(p, \omega)}{[\omega - \varepsilon_p - \text{Re} \Sigma(p, \omega)]^2 + [\Gamma'(p, \omega)/2]^2}.
$$

(2.34)

Here \(\text{Re} \Sigma(p, \omega)\) is the part of \(\Sigma(p, z)\) which is continuous as \(z\) crosses the real axis. It is given by an expression identical with (2.33) except that

$$
\frac{1}{z \pm \omega_q - \omega'} \rightarrow \frac{1}{\omega \pm \omega_q - \omega'}
$$

where \(P\) stands for principal value. \(\Gamma'(p, \omega)\) is the discontinuity in (2.33). In \(\Gamma'\), the energy denominators in (2.33) are replaced by \(\delta\)-functions:

$$
\frac{1}{z \pm \omega_q - \omega'} \rightarrow 2\pi\delta(\omega \pm \omega_q - \omega').
$$

Since \(A(p, \omega)\) is defined in terms of \(G^r\) and \(G^c\) by (2.29b) and \(\Sigma(p, z)\) is defined in terms of \(G^r\) and \(G^c\) by (2.33), Eq. (2.34) gives us one relation between \(G^r\) and \(G^c\). The boundary condition

$$
G^c(p, \omega) = \exp(-\beta\omega) G^r(p, \omega)
$$

(2.35)
provides another relation between $G^>$ and $G^<$. Thus, we have two equations in two unknowns and we may solve for $G^>$ and $G^<$.  

III. Specialization to a Metal

At this point, I would like to quote two rather surprising facts about the approximation described above:

(a) This approximation is exact in a normal (nonsuperconducting) metal if the correct phonon energy spectrum is employed and the electron-electron interactions are neglected.

(b) The highly nonlinear equations we have written down are exactly soluble for a metal.

These statements have been proved by Migdal (8). Before outlining Migdal's proof, I should point out that these statements are very closely related to the well-known fact that the adiabatic approximation (9) gives a correct description of the electron-phonon interaction in metals. Therefore, I and II really tell us that our Green's function approximation is a correct restatement of the adiabatic approximation.

In order to see the essence of Migdal's arguments, let us consider the diagrammatic expansion for the self-energy in a power series in $G$ and $V$. The terms we have considered in the self-energy correspond to the diagrams

\[
\Sigma_1(1, 1') = \delta(1 - 1') \left[ -i \int d2\nu(1 - 2) G(2, 2) \right] \quad (3.1)
\]

and

\[
\Sigma_2(1, 1') = iV(1, 1') G(1, 1'). \quad (3.2)
\]
Here the \( \rightarrow \rightarrow \) line stands for a \( G \) and \( \sim \sim \) for the interaction \( V \). The first of these terms vanishes and the next is the one we have taken into account.

The next order terms, which we have neglected are:

\[
\Sigma_2(1, 1') = i\delta V(1, 1') G(1, 1') \tag{3.3}
\]

\[
\delta V(1, 1') = -i \int V(1, 2) G(2, 2') V(2, 1') \, d2' \, d2
\]

and

\[
\Sigma_4(1, 1') = i^2 \int d2 \, d2' \, V(1, 2) G(1, 2') G(2', 2) G(2, 1') V(2, 1'). \tag{3.4}
\]

The substance of point 1 in the Migdal paper is that terms like \( \Sigma_4 \) are negligible while terms like \( \Sigma_2 \) serve to shift the phonon energies in the \( V(1, 1') \) of \( \Sigma_2 \) from the bare phonon energies to the energies of the fully interacting phonon system. The manner in which \( \Sigma_2 \) serves to shift the energies is, I think, clear. However, some further discussion is necessary in order to see why \( \Sigma_4 \) is small.

A crucial fact in this discussion is that only electrons with very small values of \( \varepsilon_r (\varepsilon_r \ll \mu) \) play any role in the electron-phonon interaction. The reason for this limitation lies in the fact that the phonon energies are all quite small, in fact,

\[
(\varepsilon_r/\mu) \sim \sqrt{m/M} \ll 1 \tag{3.5}
\]

where \( m \) is the electronic mass and \( M \) the ionic mass. Therefore, an electron which initially lies quite close to the Fermi surface must always
remain quite close to \( \epsilon_p \approx 0 \). If all electronic momenta must be very close to

\[
p = p_F = \sqrt{2m\mu}.
\]

This puts considerable restrictions on the momentum integrals which are to be done in evaluating \( \Sigma_1 \). In fact, this phase space restriction leads to \( \Sigma_1 \) being smaller than \( \Sigma_2 \) by a factor of order \( \sqrt{m/M} \). (We assume that the dimensionless coupling constant \( |v_0|^2 m p_F^2 / 2\hbar \) is of order unity.)

In this way, Migdal proved that the approximation that we discussed in the last two sections was an exact perturbation theoretic expansion of the self-energy to order \( (m/M)^{1/2} \). Migdal apparently did not publish this proof for several years because it obviously contained a grave defect—it appears to prove that the electron-phonon interaction has nothing to do with superconductivity. The trouble is that, for a superconductor, perturbation theory does not converge. Hence this whole perturbation theoretic argument is quite wrong for a superconductor.

We shall return to the discussion of the superconductor later. For now let us consider Migdal’s second point. We can solve (2.34) for \( A(p, \omega) \). To effect this solution we examine (2.33) and notice that \( \Sigma(p, z) \) is a very slowly varying function of \( p \), but a rapidly varying function of \( z \). The rate of variation in \( z \) is measured by \( \omega_0 \), but the rate of variation in \( \epsilon_p = (p^2/2m) - \mu \), is measured by \( \mu \). Because of his slow variation in \( p \), we can simply set \( p = p_F \) in Eq. (2.33) and write (2.34) as

\[
A(p, \omega) = \frac{1'(\omega)}{[\omega - \epsilon_p - \text{Re} \Sigma(\omega)]^2 + [1'(\omega)/2]^2} \tag{3.6}
\]

with

\[
\text{Re} \Sigma(\omega) = \text{Re} \Sigma(p_F, \omega)
\]

\[
1'(\omega) = \text{Re} 1'(p_F, \omega).
\]

Let us observe that (3.6) implies

\[
\int \frac{d\epsilon}{2\pi} A(p, \omega) = 1. \tag{3.7}
\]

Equation (3.7) is crucially important because a huge variety of physical quantities depend upon the integral (3.7). Because this integral is in-
dependent of the strength of the electron-phonon interaction, a whole host of calculations are immensely simplified. The physical meaning of (3.7) can be roughly appreciated if we notice that the electron-phonon interaction serves to scatter electrons only over the very narrow range of momenta

$$\left| \frac{p^2}{2m} - \mu \right| \lesssim \omega_q .$$

That is to say the interaction produces a quasi-particle state by redistributing the electron momenta and superposing states of slightly different electronic momentum. However, when we examine a sum over all $p$ (or equivalently, all $\varepsilon$) all this redistribution becomes irrelevant so that the integral in (3.7) is quite independent of the strength of electron-phonon interaction.

Equation (3.7) can be used to simplify the expression (2.33) for $\Sigma(p, \varepsilon)$. This simplification is effected by writing

$$\int \frac{d^3p}{(2\pi)^3} = \int d\varepsilon_p \int \frac{d\Omega}{4\pi} \frac{mp}{2\pi^2} \approx \frac{mp_F}{2\pi^2} \int d\varepsilon \int \frac{d\Omega}{4\pi} \tag{3.8}$$

where

$$\Omega = \hat{p} \cdot |p| \approx p |p_F| .$$

We shall employ the abbreviation

$$N(0) = \frac{mp_F}{2\pi^2} \tag{3.9}$$

for the (unperturbed) density of states in energy at the Fermi surface. Then by using the relations

$$G^+(p, \omega) = \left[ 1 + f(\omega) \right] A(p, \omega)$$
$$G^-(p, \omega) = f(\omega) A(p, \omega),$$

we can rewrite Eq. (2.33) as

$$\Sigma(z) = N(0) \int d\varepsilon \int \frac{d\Omega}{4\pi} \left| \nu \right|^2$$
$$\cdot \left\{ 1 + f(\omega') \right\} \left[ \frac{N_q}{z + \omega_q - \omega'} + \frac{N_q + 1}{z - \omega_q - \omega'} \right] \tag{3.10}$$
\[ + f(\omega') \left[ \frac{N_q}{2 - \omega - \omega'} + \frac{N_q + 1}{2 + \omega - \omega'} \right] \]
\[ \cdot \int \frac{d\epsilon'}{2\pi} A(\epsilon', \omega') \cdot q = p_p \mid \Omega - \Omega' \mid \] (3.10 cont.)

Since the \( \epsilon' \) integral of \( A \) is unity according to (3.7), we have reduced the evaluation of \( \Sigma(z) \) and hence \( A(p, \omega) \) to quadratures. Schrieffer and Englesberg (10) have done these integrals at zero temperature and examined the form of \( A \).

However, for most purposes it is not even necessary to know \( A \): the sum rule (3.7) is quite sufficient for obtaining a variety of useful results. For example, consider a tunneling experiment: We can rewrite expression (1.12) for the tunneling rate as

\[
\text{net} \quad \frac{dN}{dt} = 2 \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \\
\cdot | T(p, p') |^2 A_A(p, \omega) A_B(p', \omega') \\
\cdot 2\pi \delta(\omega - \omega' + eV) [f(\omega) - f(\omega')] .
\] (3.11)

(The factor 2 is a result of the sum over electron spins.) Then we can use the fact that \( T(p, p') \) depends only very weakly upon \( \epsilon \) and \( \epsilon' \) to do the \( \epsilon \) and \( \epsilon' \) integrals in (3.11). The result

\[
\text{net} \quad \frac{dN}{dt} = 2[N(0)]^2 2\pi \int \frac{d\Omega'}{4\pi} \int \frac{d\Omega'}{4\pi} | T(p_p, \Omega_p, p_p, \Omega') |^2 \\
\cdot \int d\omega \int d\omega' \delta(\omega - \omega' + eV) [f(\omega) - f(\omega')] \\
= 2[N(0)]^2 2\pi \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} | T(p_p, \Omega_p, p_p, \Omega') |^2 eV
\] (3.12)

is very interesting because all effects of the electron-phonon interaction have completely cancelled out.

Notice, incidentally, the complete failure of the conventional quasiparticle picture which gives the effective density of states in energy as

\[ N(0) \frac{d\epsilon}{dE} \] (3.13)
with \( E \) given by

\[
E_p = E_p + \Sigma(p, E_p).
\]

(3.14)

Even for regions of \( E \) in which \( \Sigma(p, E) \) is completely real so that the quasi particles are infinitely long-lived expression (3.13) is irrelevant. The relevant density of states for tunneling is the simpler expression \( N(0) = mp_f/2\pi^2 \).

\[ \text{IV. The Superconductor} \]

All of the arguments of the previous lectures can be generalized to the case of the superconductor. This generalization requires the use of the additional propagators

\[
F(1, 2) = \frac{1}{i} \left< T[\psi_t(1) \psi_t(2)] \right>
\]

(4.1)

and

\[
F^*(1, 2) = \frac{1}{i} \left< T[\psi_t^*(1) \psi_t^*(2)] \right>,
\]

(4.2)

which respectively describe processes in which a bound pair breaks up so that two particles may be removed from the system and processes in which two particles added to the system form a bound pair. (The formulation given here is appropriate in the case in which the pairs are bound in an \( s \) state of zero total spin. In more complex situations, more correlation functions need be considered. See the work of R. Balian in this volume.)

It is important that \( F \) and \( F^* \) be considered handled in exactly the same way as G. Nambu\(^4\) (11) has introduced a simple trick for handling \( G, F, \) and \( F^* \) at the same time. He begins with the spinor fields

\[
\Psi(1) = \begin{pmatrix} \psi_t(1) \\ \psi_t^*(1) \end{pmatrix}; \quad \Psi^*(1) = \begin{pmatrix} \psi_t^*(1') \\ \psi_t(1') \end{pmatrix}
\]

(4.3)

and notices that the basic Hamiltonian may be rewritten in terms of

\(^4\) Nambu's formalism provides a very convenient technique for extending the B.C.S. theory (12) to the case of retarded interactions.
these fields. For example, the kinetic energy is:

\[ K.E. = \int dr \left\{ \left( \frac{V \cdot V'}{2m} - \mu \right) \left[ \psi^*_+(r') \psi_+(r) + \psi^*_-(r') \psi_-(r) \right] \right\}, \]

\[ = \int dr \left\{ \left( \frac{V \cdot V'}{2m} - \mu \right) \left[ \psi^*_+(r') \psi_+(r) - \psi^*_+(r) \psi_+(r) + \delta(r - r') \right] \right\}. \]

The \( \delta \)-function term is quite irrelevant since it is just a number added to the Hamiltonian. When this term is neglected the kinetic energy can be written in terms of an inner product of the spinors

\[ K.E. = \int dr \left\{ \left( \frac{V \cdot V'}{2m} - \mu \right) [\psi^*_+(r') \tau_3 \psi_+(r)] \right\}. \tag{4.4} \]

Here \( \tau_3 \) is the standard Dirac spin matrix

\[ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Similarly, the interaction term can be written in terms of \( \Psi^+ \) and \( \Psi' \) as

\[ H_{\rho'} = \int dr \left[ \psi^*_+ \rho \psi_+ \exp (iq \cdot r) + \rho^*_+ \psi_+ \psi_+ \psi_+ \exp (-iq \cdot r) \right] \]

\[ = \int dr \left[ \psi^*_+ \rho \psi_+ \psi_+ \psi_+ \exp (-iq \cdot r) \right] [\psi^*_+(r') \tau_3 \psi_+(r)]. \tag{4.5} \]

The properties of the superconductor are described by a spinor Green's function

\[ g(1, 1') = \frac{1}{i} \langle T[\Psi(1') \Psi^+(1)] \rangle \]

\[ = \begin{pmatrix} G(1, 1') & F(1, 1') \\ F^*(1, 1') & -G(1', 1) \end{pmatrix}. \tag{4.6} \]

Since this spinor function obeys the same boundary condition as \( G \), it has exactly the same formal properties as \( G \). In particular, we can notice that it can be expanded in Fourier series with matrix Fourier coefficients \( g(p, z) \) where
THE ELECTRON-PHONON INTERACTION IN METALS

\[ g(p, z) = \int \frac{d\omega}{2\pi} \frac{a(p, \omega)}{z - \omega} = \int \frac{d\omega}{2\pi} \frac{g^\ast(p, \omega) + g^\ast(p, \omega)}{z - \omega}. \] (4.7)

The two pieces of \( g \) are again related by

\[ g^\ast(p, \omega) = \exp \left( -\beta \omega \right) g^\ast(p, \omega). \] (4.8)

Equivalently, we can write

\[ g^\ast(p, \omega) = f(\omega) a(p, \omega) \]
\[ g^\ast(p, \omega) = [1 - f(\omega)] a(p, \omega), \]

where \( a(p, \omega) \) is now a matrix.

We can express

\[ g(p, z) = \frac{1}{z - \varepsilon_p - \sigma(p, z)} = \int \frac{d\omega}{2\pi} \frac{a(p, \omega)}{z - \omega} \] (4.10)

where the matrix \( \sigma(p, z) \) is expandable in exactly the same kind of perturbation theory as in the normal state. The only difference lies in the fact that the matrix \( g \) replaces the scalar \( G \) and a factor of \( \tau_3 \) appears at each vertex. Thus, in lowest order

\[ \sigma(1, 1') = \]
\[ = i V(1 - 1') \tau_3 g(1, 1') \tau_3 \] (4.11)

or

\[ \sigma(p, z) = \int \frac{d^3 p'}{(2\pi)^3} \frac{d\omega'}{2\pi} | v_p |^2 \]
\[ \cdot \left\{ \tau_3 g^\ast(p', \omega') \tau_3 \left[ \frac{N_q}{z + \omega_q - \omega'} + \frac{N_q + 1}{z - \omega_q - \omega'} \right] \right\} \] (4.12)

Before we go any further, we should notice that, as in the normal metal, \( \sigma(p, z) \) can be considered to be independent of \( p \). Hence, we write \( \sigma(p, z) \approx \sigma(p_F, z) = \sigma(z) \).

In order to solve \( g(p, z) \) we guess the form of the answer for \( \sigma(p, z) \).
Our guess, which we verify below, is that $\sigma(p, z)$ contains no contribution proportional to $\tau_3$ and none proportional to $\tau_2$, but only terms proportional to the unit matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Thus we write

$$\sigma(z) = \sigma_0(z) 1 + \sigma_1(z) \tau_1. \quad \text{(4.13)}$$

Actually it is slightly more convenient to express $\sigma(p, z)$ in terms of the variables

$$Z(z) = [1 - \sigma_0(z)/z]$$

$$Z(z) \Delta(z) = \sigma_1(z) \quad \text{(4.14)}$$

with this redefinition of the variables, Eq. (4.10) becomes

$$g(p, z) = \frac{1}{Z(z) - \varepsilon \tau_3 - \Delta(z) Z(z) \tau_1}$$

$$= \frac{z Z(z) + \varepsilon \tau_3 + \Delta(z) Z(z) \tau_1}{Z^2(z) \left[ z^2 - J(z) \right] - \varepsilon^2} \quad \text{(4.15)}$$

and

$$\int \frac{d\varepsilon}{2\pi} g(p, z) = \frac{z + \Delta(z) \tau_1}{\sqrt{-z^2 + \left[ \Delta(z) \right]^2}} = \int \frac{d\varepsilon}{2\pi} \int \frac{d\omega}{2\pi} \frac{a(p, \omega)}{z - \omega}. \quad \text{(4.16)}$$

Here, the sign of the square root is chosen so that a $z \to \infty$ below the real axis $\sqrt{-z^2} \to -iz$, while as $z \to \infty$ above the real axis $\sqrt{-z^2} \to +iz$. There is a branch line in the definition of the square root along the real axis. The $\varepsilon$ integral of $a(p, \omega)$ is now easily computed in terms of the discontinuity of Eq. (4.16) as $z$ crosses the real axis. This gives

$$\int \frac{d\varepsilon}{2\pi} a(p, \omega) = \text{Re} \frac{\omega + \Delta(\omega) \tau_1}{\sqrt{\omega^2 - \left[ \Delta(\omega) \right]^2}} \quad \text{(4.17)}$$

$^1$ The first application of this $\varepsilon$ integration trick to the superconductor appeared in P. Morel and P. W. Anderson (13).
where the square root is defined by the statement
\[
\lim_{\omega \to \infty} \left\{ \sqrt{1 - \frac{\omega^2}{\omega'}^2} \right\} = 1.
\]  \hspace{1cm} (4.12)

Now we return to Eq. (4.12). We make the substitution \( \varphi \equiv (1 - f) a \), \( g^x = f a \), and write the momentum integral as
\[
\int \frac{d^3 p'}{(2\pi)^3} \to N(0) \int d\omega' \int \frac{d\omega}{4\pi},
\]

after the \( \epsilon' \) integral is performed with the aid of Eq. (4.17), we find
\[
\varphi(z) = z[1 - Z(z)] + Z(z) \Delta(z) \tau_1
\]
\[
= \int d\omega' \Re \left[ \frac{\omega' \left( 1 - \Delta(\omega') \right) \tau_1}{\sqrt{\omega'^2 - [\Im(\omega')]^2}} \right] K(z, \omega')
\]  \hspace{1cm} (4.18)

where
\[
\begin{aligned}
K(z, \omega') &= N(0) \int \frac{d\omega}{4\pi} \left| \nu_\omega \right|^2 \\
&= \left\{ \frac{N_z[1 - f(\omega')]}{z + \omega - \omega'} + \frac{N_y f(\omega')}{Z - \omega} \right\}_{\omega - \omega'} \\
&+ \left\{ \frac{N_x[1 - f(\omega')] + N_y f(\omega')}{z + \omega - \omega'} \right\}_{\omega = \omega'}
\end{aligned}
\]  \hspace{1cm} (4.19)

We see that our original assumption, Eq. (4.13) about the form of \( \varphi(z) \) is indeed justified.

It is interesting to note that Eq. (4.18) may be reduced to an integral equation for \( \Delta \) alone since
\[
Z(z) = 1 - \frac{1}{z} \int d\omega' \omega' \Re \left[ \frac{1}{\sqrt{\omega'^2 - [\Im(\omega')]^2}} \right] F(z, \omega')
\]  \hspace{1cm} (4.20)

so that
\[
\Delta(z) = - \int d\omega' \left\{ K(z, \omega') \right\}
\]  \hspace{1cm} (4.21)

\[
\left\{ \Re \frac{\Delta(\omega')}{\sqrt{\omega'^2 - [\Im(\omega')]^2}} - \frac{\Delta(z)\omega'}{z} \Re \frac{1}{\sqrt{\omega'^2 - [\Im(\omega')]^2}} \right\}.
\]

An equation essentially similar to this one has been solved by J.R. Schrieffer et al. (14) in the zero temperature limit. The only modification
in this work beyond the formulas indicated here is the appearance of a pseudopotential in $K(z, \omega')$ which is introduced as a representation of the coulomb potential between electrons.

This solution can be used to describe the tunneling characteristics of a junction between a normal metal and a superconductor. We again use Eq. (3.11)

$$\frac{dN}{dt} \bigg|_{A \rightarrow N} = 2[N(0)] \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} \left| T(p, p') \right|^2$$

for the case in which $A$ is a superconductor and $B$ a normal metal. Just as before, we employ

$$\int d\epsilon' A_B(\epsilon', \omega') = 2\pi.$$

For the superconductor, Eq. (4.16) implies

$$\int dt \alpha(\epsilon, \omega) = 2\pi \Re \left\{ \frac{Z + \mathcal{J}(z) \tau_1}{\sqrt{z^2 \left[ \mathcal{J}(z) \right]^2}} \right\}_{r=n}^1.$$

(4.22)

Since the scalar $A(p, \omega)$ is the upper left-hand corner of matrix $a(p, \omega)$

$$\int dt A_A(\epsilon, \omega) = 2\pi \Re \frac{\omega}{\sqrt{\omega^2 - \left[ J(\omega) \right]^2}}.$$

(4.23)

This expression for the density of states was first derived by Schieffer et al. (14).

In this way, we find that the tunneling rate is given by

$$\frac{dN}{dt} \bigg|_{A \rightarrow N} = 4\pi[N(0)]^2 \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} \left| T(p, p') \right|^2$$

$$\int d\omega \int d\omega' \delta(\omega - \omega' + eV) \left| f(\omega) - f(\omega') \right|$$

$$\Re \frac{\omega}{\sqrt{\omega^2 - \left[ J(\omega) \right]^2}}.$$
Consequently, the quasi-particle picture gives an effective tunneling density of states as

\[
N(0) \left[ \frac{E}{\sqrt{E^2 - [\Delta(E)]^2}} - \Delta(E) \frac{d\Delta(E)}{dE} \frac{1}{\sqrt{E^2 - [\Delta(E)]^2}} \right].
\]  

(4.28)

Whenever \( \Delta(E) \) depends upon \( E \) the quasi-particle picture gives a density of states different from the exact expression, Eq. (4.27). Therefore, the quasi-particle picture fails whenever \( \Delta(E) \) depends upon \( E \).

REFERENCES