Transport Coefficients near the $\lambda$-transition of Helium*

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Transport coefficients are computed near the $\lambda$-transition of helium by considering processes in which one transport mode breaks up into several. The conclusions are that first sound damping diverges as $|\epsilon|^{-1}$, second sound damping as $|\epsilon|^{-1/2}$ and thermal conductivity as $|\epsilon|^{-1/3}$ with $\epsilon = (T - T_\lambda)/T_\lambda$ for low frequencies. These results were first obtained via the dynamical scaling idea by Ferrell and coworkers. Less singular results are found at higher frequencies.

1. INTRODUCTION

One main line of investigation of transport coefficients near critical points is the dynamical scaling method of Ferrell et al. (1)–(3) and of Halperin and Hohenberg (4). This approach asserts that there is a single characteristic complex frequency $\omega(q)$ which determines the majority of the dynamical behavior near the critical point and further that this frequency is a homogeneous function of $q$ and the thermodynamic variables which go to zero at the critical point. Consequently, this work may be considered to be an extension into dynamical behavior of the static scaling theories proposed by Widom (5) and others (6, 7).

Another thread of theories of dynamics near critical points is due to Fixman (8) and Kawasaki (9). These authors look at the breakup of individual transport modes into several rather long wave-length modes. More recently, the present authors (10) and also Kawasaki (11) have incorporated scaling ideas into this analysis as a method of estimating the rates of breakup. This analysis agrees in part with the results of the dynamical scaling point of view. However, according

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to reference (10), the scaling approach must be modified to include the existence of several characteristic frequencies which have different characteristic sizes—i.e. they scale differently—and which mix together to determine the critical dynamics. In particular, reference (10) describes the interaction between three characteristic frequencies which are the thermal relaxation rate, the viscous relaxation rate, and the sound frequency at wave vectors of the order of an inverse coherence length.

In the present paper, the methods of reference (10) are extended to the discussion of the \( \lambda \)-transition of \( ^4 \)He. This was the situation which was first studied by the dynamical scaling method (1-3). Ferrell and coworkers applied the scaling ideas to interconnect second sound and thermal conduction. We consider two characteristic modes: second sound and first sound. Our results are very similar to those of references (1-3).

The next section describes the formulation of our approach to this problem. Then Section 3 considers the estimation of necessary matrix elements while Section 4 estimates the transport coefficients. The final section lists our conclusions.

2. FORMULATION

The basic quantum mechanical operators describing the hydrodynamics near the \( \lambda \)-transition are the density, \( \rho(q) \), the entropy density (12) \( s(q) \), the momentum density \( g(q) \) and the superfluid velocity (13) \( v_s(q) \). For small \( q \) the fluctuations in these operators are described by thermodynamics as

\[
\langle \rho(q) \rho(-q) \rangle = \rho T \left( \frac{\partial \rho}{\partial p} \right)_T, \\
\langle s(q) s(-q) \rangle = \rho C_p, \\
\langle g(q) g(-q) \rangle = \rho T, \\
\langle v_s(q) v_s(-q) \rangle = T/\rho_s, 
\]

in units in which the Boltzmann constant is one. Both theory (14) and experiment (15) give \( \rho_s \), going to zero as \( (T_\lambda - T)^{2/3} \) based upon the logarithmic divergence of \( C_p \) (16).

In a correlation function \( s(q) \) and \( \rho(q) \) behave quite similarly. Both the specific heat at constant pressure and \( (\partial \rho/\partial p)_T \) diverge logarithmically near the phase transition (16). Consequently, inside a correlation function \( \rho(q) \) and \( s(q) \) behave similarly. Therefore we write (17)

\[
\frac{\rho(q)}{\rho} \sim \frac{s(q)}{s} \sim T \frac{\partial}{\partial T} \bigg|_p, 
\]
The transport analysis of reference (10) is based upon the use of quasi-classical "creation" operators for transport modes. For small $q$ pointing in the $x$-direction, these creation operators for first and second sound are

$$a_1(q) = \frac{b_1(q)}{\sqrt{2}} \pm \frac{g_s(q)}{(2\rho T)^{1/2}},$$

$$a_2(q) = \frac{b_2(q)}{\sqrt{2}} \pm [v_s(q)]_\| (\rho_\|/2T)^{1/2},$$  \hfill (3)

where the $\pm$ sign refers to the two possible directions of propagation. Terms explicitly of the order of $\rho g(q)$ have been dropped in Eq. (3).

Then $b_1$ and $b_2$ are linear combinations of the two basic scalar operators $\rho$ and $s$. For example, low frequency second sound has

$$b_2(q) = [s(q)]/[\rho C_s]^{1/2}.$$

The transport coefficient below the $\lambda$-point are, in the notation of Khalatnikov (18) $\zeta_1$, $\zeta_2$, $\zeta_3$, $\eta$ and we call the thermal conductivity $\lambda$. These may be calculated via Kubo formulas (19) as, for example (20),

$$\zeta_i(q,s)q^2 = (1/T) \int_0^\infty dt \ e^{-i\omega} \left< \left\{ \frac{d}{dt} a_i(q,t) \right\} P_{i+}(t) \left[ P_-(t') \frac{dg(q,t')}{dt'} \right]_{t'=0} \right>. \hfill (4)$$

Here $\left\langle \right|$ represents a typical state of the system at the appropriate temperature and pressure while $P_i$ is a projection operator which projects out states of the form $a_i(q)|\rangle$, where $a_i$ is any one of the four basic operators which create a single long-wavelength mode

$$P_i = 1 - \sum_{i=1}^2 a_i(-q)|\rangle \langle a_i(q). \hfill (5)$$

Here $i = 1$ refers to first sound, $i = 2$ to second sound. A sum over two propagation directions is implicitly included here and below.

To evaluate Eq. (4), replace the unit operator in the expectation value by a sum over transport modes as

$$1 = \frac{1}{2!} \sum_{i=1}^2 \sum_{i=1}^2 \sum_{q'q} e^{-[s_i(q') + s_i(q)]t} a_i(q',t) a_i(q',t) \left< \left| a_i(-q') a_i(-q') + \cdots . \right. \right.$$

Then Eq. (4) emerges as

$$\zeta_i(q,s)q^2 = \frac{1}{2T} \sum_{i=1}^2 \sum_{i=1}^2 \sum_{q'q} \frac{M(v_s) M^*(q)}{s + s_i(q') + s_i(-q' - q')} \hfill (6)$$
where the matrix elements are given by
\[
M_{\ell_0}^{\ell_1}(v_s) = \left\langle \left| \frac{d\nu_s(q, t)}{dt} \right|_{t=0} P_q a_\ell(q') a_{\ell'}(-q - q') \right\rangle. \tag{7}
\]

Precisely similar forms arise for $\ell_0 + \ell_0$—which has the matrix elements $M(q)^4$—for $\ell_0$—which has as matrix elements $M(v_s)^4$—and $\lambda$—which involves a product of matrix elements of the form $M(s)^4$.

3. ESTIMATION OF MATRIX ELEMENTS

To estimate the matrix element, (7), use the equation of motion of $v_s$, $m \frac{dv_s}{dt} = -\nabla \mu$ to obtain
\[
M_{\ell_0}^{\ell_1}(v_s) = \left\langle (-i q/m) \mu(q) P_q a_\ell(q') a_{\ell'}(-q - q') \right\rangle. \tag{8}
\]

When $q$ and $q'$ are less than or of the order of the inverse coherence length, it is probably appropriate to replace the matrix element by its thermodynamic value, that is its value as $q'$ and $q$ become small. There are several terms in the matrix element of the same apparent size. If one assumes that the leading singular terms do not cancel one can replace each term in the matrix element by a typical leading singular term, as for example,
\[
a_\ell(q') \rightarrow \frac{s(q')}{(\rho C_p)^{3/2}},
\]
\[
a_{\ell'}(-q - q') \rightarrow \frac{s(-q - q')}{(\rho C_p)^{3/2}},
\]
\[
P_q \rightarrow \frac{s(-q')}{\rho C_p}.
\]

Then, one finds
\[
M_{\ell_0}^{\ell_1}(v_s) \sim iq \left\langle \mu(q) s(-q') \left\langle s(q) s(-q - q') s(q') \right\rangle \right\rangle \frac{m(\rho C_p)^2}{\rho C_p}
\sim iq \left( \frac{\partial \mu}{\partial T} \right)_p \left( \frac{\partial C_p}{\partial T} \right)_p \frac{T^2}{(\rho C_p)^3 m}.
\]

The leading singularity comes from $(\partial C_p/\partial T)_p$ which diverges linearly in $\epsilon \equiv (T - T_\lambda)/T_\lambda$. Since the scaling arguments are not good enough to give logarithmic terms correctly, we simply estimate this matrix element as $| \epsilon |^{-1}$ times an appropriate slowly varying dimensional factor. Since the scaling analysis gives $| \epsilon |^{-1} \sim \xi^{3/2}$, we can write
\[
M_{\ell_0}^{\ell_1}(v_s) \sim iq c_\ell \epsilon(T/p_\lambda)^{1/2} \xi^{3/2} \quad \text{for} \quad q \lesssim \xi^{-1} \quad q' \lesssim \xi^{-1}. \tag{9a}
\]
The result would have been just the same had we replaced $a_i$ and $a_j$ by density operators.

Eq. (9a) gives the matrix element for superfluid flow to turn into two sound waves. A similar matrix element can be computed for normal flow. Exactly the same argument works for the normal flow and gives the same result, namely,

$$M_{\text{SW}}^{ij}(\xi) \sim iqc_s(\rho T)^{1/2} \xi^{3/2}. \quad (9b)$$

Finally, we estimate the matrix element for the breakup of a heat flow. For heat-flow

$$M_{\text{HE}}^{ij}(s) = \langle |(d/dt)s(q) P_{s}(a_i(q')) a_f(-q - q')| \rangle. \quad (10)$$

The time derivative can be replaced by $-iqj'(q)$ where $j'$ is the heat current. Then the projection operator gives no contribution at all above $T_s$. Consequently, to obtain an estimate of the matrix element above and below $T_s$ we replace $P$ by unity in Eq. (10) and write

$$\frac{M_{\text{HE}}^{ij}(s)}{-iq} = \langle | j'(q) a_i(q') a_f(-q - q')| \rangle. \quad (11)$$

When the creation operators in Eq. (11) refer to second sound, the matrix element can be estimated rather directly. The $a_i(q)$ is replaced by the vector part of the creation operator

$$a_i(q') \rightarrow (\rho_s/T)^{1/2} v_s(q')$$

while the scalar part is replaced by something of the order of magnitude of a normalized entropy density

$$a_f(-q - q') \rightarrow \frac{s(-q - q')}{(\rho_s/C_p)^{1/2}}.$$ 

Furthermore, since in the superfluid state $j'$ contains a term like $(s/p) \rho_s v_s$, we assume that the fluctuations in $j'$ are similar to the fluctuations in $(s/p) \rho_s v_s(q)$. Therefore, we write

$$\frac{M_{\text{HE}}^{ij}(s)}{-iq} \sim \frac{(\rho_s s/p)}{(\rho_s C_p)^{1/2}} \langle | v_s(q)(\rho_s/T)^{1/2} v_s(q') s(-q - q')| \rangle$$

as $q$ and $q'$ go to zero, this matrix element reduces to

$$\frac{M_{\text{HE}}^{ij}(s)}{-iq} \sim \frac{\rho_s^{3/2}s}{(\rho_s C_p T)^{1/2}} T \frac{\partial}{\partial T} \langle | v_s(q) v_s(-q)| \rangle$$

$$\sim \frac{\rho_s^{3/2}}{(\rho_s C_p T)^{1/2}} e^{1} \rho_s^{1} \sim \rho_s^{1} e^{1} \left( \frac{\rho_s C_p}{T} \right)^{1/2}.$$
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Since \( \rho^{1/2} \sim \epsilon^{1/3} \sim c_z \) the result is like \( \xi^{3/2} c_z \) times appropriate dimensional factors. We write

\[
M^{(1)}_{\text{sc}}(s) \sim i q (\rho C_p)^{1/2} c_z \xi^{3/2}.
\]  

(12)

The estimate (12) for the matrix element should be good for \( T > T_\lambda \). It should be correct for \( T < T_\lambda \) provided that the singular contribution from the local equilibrium part of the projection and that just calculated do not cancel against each other.

The matrix element for a heat flow becoming two first sound waves is harder to estimate. Since \( \rho \) has only a very small singularity one might argue that \( g(q) \) has only very weak fluctuations. But this result holds only for very small \( q \). If we assume that this behavior holds true for all \( q \lesssim \xi^{-1} \) we may estimate \( M^{(1)}_{\text{sc}}(s) \) in the following manner.

If \( a_i(q) \) is replaced by the vector part of the first sound creation operator

\[
a_i(q') \rightarrow \frac{1}{(\rho T)^{1/2}} g(q')
\]

while the scalar part is again replaced by the normalized entropy density, then we obtain

\[
M^{(1)}_{\text{sc}}(s) \sim \frac{\rho (s/\rho)}{(\rho C_p)^{1/2}} \left\langle \bar{v}_i(q) \left( \frac{1}{\rho T} \right)^{1/2} g(q') s(-q - q') \right\rangle.
\]

Since at \( q = 0 \)

\[
\langle \bar{v}_i(q) g_i(-q) \rangle_{q=0} = T
\]

and since we are assuming that \( g(q) \) contains only small fluctuations for \( q \lesssim \xi^{-1} \) we may estimate that there is no singular contribution to the thermal conductivity from first sound waves in the intermediate state.

The reader should recognize that the argument which led to this estimate involved several guesses and assumptions.

4. ESTIMATES OF TRANSPORT COEFFICIENTS

The estimates of the transport coefficients are now quite straightforward. Eqs. (6) and (9) imply that, for \( q \lesssim \xi^{-1} \)

\[
\xi_2(q, s) \sim \left( \frac{\rho}{\rho_s} \right)^{1/2} \sum_{i=1}^n \sum_{j=1}^n \int_{q' \leq \xi} dq' \frac{c_2 \xi a}{s + s_i(q') + s_j(-q')}.
\]  

(13)
When $s \ll c_2 \xi^{-1}$, the main contribution comes from second sound wave intermediate states, for which

$$s_2(q') \sim c_2 q'.$$

Since the integral contributes over a volume like $\xi^{-3}$,

$$\zeta_2(q, s) \sim \left( \frac{\rho}{\rho_s} \right)^{1/2} \frac{c_2 c_1}{c_2^2} \sim |e|^{-1} \quad \text{for} \quad q \ll \xi^{-1}, \quad s \ll c_2 \xi^{-1}. \quad (14a)$$

The corresponding result for $\zeta_3$ was first derived by Ferrell et al. When the frequency is large compared with the characteristic second sound frequency but small compared with $c_1 \xi^{-1}$, the main contribution to (17) comes from first sound wave intermediate states. Then,

$$\zeta_3(q, s) \sim \left( \frac{\rho}{\rho_s} \right)^{1/2} \frac{c_1 c_2}{c_1 c_2} \sim |e|^{-2/3} \quad \text{for} \quad q \ll \xi^{-1}, \quad s \ll c_1 \xi^{-1}. \quad (14b)$$

The estimates (14) are, of course, not good enough to pick up logarithmic multiplying factors as, perhaps, $C_\rho/C_\rho$.

Since the matrix elements for $\zeta_2$ and $\zeta_3$ are identical, except for factors of $\rho$ and $(c_2/c_1)(\rho/\rho_s)^{1/2}$,

$$\frac{c_2}{c_1} \left( \frac{1}{\rho \rho_s} \right)^{1/2} \zeta_2(q, s) \sim \left( \frac{\rho \rho_s}{\rho} \right)^{1/2} \frac{c_1}{c_2} \zeta_3(q, s) \sim \zeta_3(q, s). \quad (14c)$$

According to the approximate Kubo formula, the thermal conductivity is determined by

$$\lambda(q, s)q^2 = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{q'} \frac{|M^{ij}(s)|^2}{s + s(q') + s(-q' - q')} \quad (15)$$

From the estimate, (12), the process in which the heat flow becomes two second sound waves gives

$$\lambda(q, s) \sim \int_{q' < c^{-1}} dq' \frac{\rho C_\rho c_2^2 \xi^3}{c_2 \xi^{-1}} \sim \rho C_\rho c_2 \xi$$

so that

$$\frac{\lambda(q, s)}{\rho C_\rho} \sim c_2 \xi \sim |e|^{-1/3} \quad \text{for} \quad q \ll \xi^{-1}, \quad s \ll c_2 \xi^{-1}. \quad (16)$$

This result for $\lambda$, for $T > T_A$, was first predicted by Ferrell et al. (I).
As discussed in the preceding section there appears to be no singular contribution to $\lambda$ from first sound waves.

5. CONCLUSIONS

Near the critical point, there is one length which is particularly important, the coherence length $\xi \sim |\epsilon|^{-2/3}$. But there are two characteristic frequencies $s_1^*$ and $s_2^*$ derived from first and second sound as

$$s_1^* = c_1 \xi^{-1} \sim |\epsilon|^{2/3}$$
$$s_2^* = c_2 \xi^{-1} \sim |\epsilon|.$$(17)

In the range of frequencies much higher than $s_2^*$ but comparable to $s_1^*$, the characteristic processes which determine the transport coefficients involve propagation of signals via multiple production of first sound waves. In this region, we find

$$\zeta_q \sim \frac{c_1}{c_2} (\rho p_0)^{1/2} \zeta_1 \sim \frac{\rho p_0 c_1^2}{c_2^2} \xi^{3/2}$$
$$\zeta_q \sim \rho c_1 \xi \sim |\epsilon|^{-2/3}$$

for $q \lesssim \xi^{-1}$, $s_2^* \ll s \lesssim s_1^*$. (18)

The thermal conductivity $\lambda$ apparently has no divergent part in this frequency region.

Notice that these results agree with the dynamical scaling idea. A frequency may be formed by multiplying each of the “diffusivities” in Eq. (18) by a typical inverse length squared—that is $\xi^{-2}$. Then, we find, for example, that

$$\frac{\xi}{\rho} \sqrt{\frac{c}{s}} \sim c_1 \xi^{-1} \sim s_1^*$$

(19)

of course, in this frequency range second sound is irrelevant and all transport coefficients must be determined by first sound. Consequently, it is quite reasonable to find that when one forms a relaxation rate as in Eq. (19) the rate turns out to be comparable with a typical first sound frequency.

In the lower frequency region, $s \lesssim s_2^*$, there are two processes entering, each with its characteristic frequency, so that one should not be surprised to discover that the relaxation rates have a more complex structure than that represented by Eqs. (18) and (19). In fact, we find that processes involving flow via second sound modes with wavelength of order $\xi$ give

$$\zeta_q \sim \frac{c_1}{c_2} (\rho p_0)^{1/2} \zeta_1 \sim \frac{\rho p_0 c_1^2}{c_2^2} \xi \zeta_q \sim \frac{\rho c_1^2}{c_2} \xi \sim |\epsilon|^{-1}.$$ (20)
for $q \lesssim \xi^{-1}$ and $s \lesssim s^*_2$. The result (20) holds below $T_1$. Above $T_\lambda$, $\rho s_1$ and $\rho s_2$ vanish but $\zeta_2$ still apparently diverges as $|\epsilon|^{-1}$.

In the region of frequencies defined by Eq. (20), the main contribution to heat flow seems to come from energy transport via second sound. Hence, the estimate of $\lambda$ from Eq. (16) holds in this region

$$\frac{\lambda(q, s)}{\rho C_p} \sim c_2 \xi \sim \epsilon^{-1/3} \quad \text{for} \quad s \lesssim s^*_2, \quad q \lesssim \xi^{-1}. \quad (21)$$

How can the results (20) and (21) be fit into a dynamical scaling picture? To do this, notice that in the limit as $q \to 0$, and $\rho_x \to 0$ the damping of first and second sound are defined by damping constants

$$D(q^2) = \frac{\zeta_0 + \frac{2}{3} q^2 \rho}{\rho} - q^2 + \frac{\lambda}{(C_v - C_p)} q^2$$

$$\sim \frac{\zeta_2}{\rho} q^2 \quad (22a)$$

$$D_2 q^2 \sim \frac{\lambda}{\rho C_p} q^2 + \frac{\rho s_2}{\rho^2} \left[ \zeta_2 \rho^2 - 2 \rho \xi_1 + \xi_2 + \frac{4}{3} \gamma \right] q^2. \quad (22b)$$

The low $q$ region is bounded by the condition

$$c_1 q < s^*_2$$

or

$$q \lesssim q^* = s^*_2/c_1 = \frac{c_2}{c_1} \xi^{-1} \sim |\epsilon|. \quad (23)$$

At the value of $q^*$ defined by Eq. (23), the first sound frequency, $c_1 q^*$, crosses the typical second sound frequency, $s^*_2$. At this point, we should expect a rather severe mixing of first sound with wave vector $q^*$ and second sound with wave vector $\xi^{-1}$.

When the mixing occurs, we should expect, as Ferrell et al. pointed out (3), that the first sound should relax at a rate comparable with the mixed-in second sound. In symbols

$$\frac{\zeta_2(q, s)}{\rho} \bigg|_{q = q^*} \sim s^*_2 \xi \sim \frac{c_1}{c_2} \frac{c_2}{c_1} \xi \sim |\epsilon|^{-1}. \quad (24)$$

The frequency scaling relation (24) implies

$$\frac{\zeta_2}{\rho} \sim s^*_2 \xi \sim \frac{c_1}{c_2} \frac{c_2}{c_1} \xi \sim |\epsilon|^{-1}. \quad (25)$$
Therefore, the result (20) can be understood from the point of view of frequency scaling. Ferrell et al. (1) also argued on the basis of the scaling of frequencies that the second sound wave damping constant, \( D_2 q^2 \), should be of the order of magnitude of \( s_x^2 \) for \( q \sim \xi^{-1} \) and from this predicted that

\[
D_2 \sim s_x^2 \xi^2 \sim c_x \xi \sim | \epsilon |^{-1/3}.
\]  

(26)

From the expressions (20) and (21) for the transport coefficients in the low frequency region and Eq. (22b) for \( D_2 \) one can see that our calculation reproduces this scaling law result.

In addition to the divergences described here, all the transport coefficients should contain additive nondivergent terms arising from high frequency intermediate states. These additive terms should vary slowly in the critical region.

The result (21) for the thermal conductivity seems to agree with the results of Archibald, Mochel, and Weaver (21) who measured \( \lambda \) just above the phase transition, at least if one does not get too close to \( T_\lambda \), and also with the earlier experiment of Kerrisk and Keller (22) as discussed by Ferrell et al. (3).

According to the discussion of Ferrell et al. (5), the experimental results of Chase (23) and the more recent results of Barmatz and Rudnick (24) on \( D_1 \) below \( T_\lambda \) agree with the scaling estimate (20) and the assumption that \( D_1 \) changes character at \( s = s_x^2 \sim | \epsilon | \). However for \( T > T_\lambda \) the work of Chase (23) and Barmatz and Rudnick (24) apparently shows that \( D_1 \sim (T - T_\lambda)^{-1/2} \). This apparent asymmetry in properties above and below \( T_\lambda \) is very hard to understand on the basis of a theory such as the one just presented and represents a serious blow to it.

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REFERENCES


12. In this and in several other matters, we follow the notation of L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) 24, 419 (1963).


17. Similar questions are discussed in Reference (10).


20. The expressions of transport coefficients in terms of current-current correlation functions have been derived for the case of superfluid helium by Hohenberg and Martin, Reference (13).


