Fundamentals of Statistical Physics

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text:
Statistical Physics,
Statics, Dynamics, Renormalization
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I also referred often to Wikipedia and found it accurate and helpful.
## Course Outline

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Part I: Once over lightly

Concepts which specifically belong to statistical physics
Interesting Physical Science Advances have a Major Statistical Component
Probabilities: One die
Quantum Stat Mech
Classical Stat Mech
Averages from Derivatives
Thermodynamics
From Quantum to Classical: The Ising model
Degenerate Distributions
Thermodynamic Phases
Phase Transitions
Random Walk
Brownian Dynamics
Big Words
Where do we come from?

Undergraduate Institution:
Concepts which specifically belong to statistical physics:
Not in quantum mechanics or in Classical Mechanics

• Temperature
Interesting Physical Science Advances have a Major Statistical Component

Bekenstein-Hawking: entropy of black holes

Fluctuation spectrum of 3 degree kelvin background radiation

Bell’s theorem: statistics of quantum measurements

Source of complexity in the universe

Probabilities of hearing from civilizations elsewhere in universe

Why do markets crash?

Time Reversal Invariance: Nature of Irreversability

Probabilities of major earth-asteroid collision

Probabilistic interpretation of quantum mechanics and of wave functions.

Is our universe likely?
Part 2. Start with Probabilities: Dice

number of times $\alpha$ turns up = $N_\alpha$; total number of events $N$

probability of choosing a side with number $\alpha = \rho_\alpha$

total probability = 1 $\Rightarrow$

$\sum_\alpha \rho_\alpha = 1$  \hspace{1cm} i.1

$r_\alpha$ = relative probability of

fair dice $\Rightarrow$ all probabilities equal

average number on a throw =

general rule: To calculate the average of any function $f(\alpha)$ that gives the probability that what will come out will be $\alpha$, you use the formula

$\langle f(\alpha) \rangle = \sum_\alpha f(\alpha) \rho_\alpha$  \hspace{1cm} i.3

Do we understand what this formula means?? How would we describe a loaded die? An average from a loaded die? If I told you that $\alpha = 2$ was twice as likely as all the other values, and these others were all equally likely, what would be the probability? What would we have for the average throw on the die?
Part 2. Start with Probabilities: Dice

number of times $\alpha$ turns up = $N_{\alpha}$; total number of events $N$

probability of choosing a side with number $\alpha = \rho_{\alpha}$ $\rho_{\alpha} = N_{\alpha}/N$ i.1

total probability =1 --> $\sum_{\alpha} \rho_{\alpha} = 1$ i.2

$\rho_{\alpha}$ = relative probability of event $\alpha$. e.g. for fair dice $\rho_{\alpha} = \text{const}$ $z = \sum_{\alpha} r_{\alpha}$ $\rho = r_{\alpha}/z$

fair dice --> all probabilities are equal --> $\rho_{\alpha} = 1/6$ for all values of $\alpha$

average number on a throw = $<\alpha> = \sum_{\alpha} \rho_{\alpha} \alpha = 3.5$

general rule: To calculate the average of any function $f(\alpha)$ that gives the probability that what will come out will be $\alpha$, you use the formula

$<f(\alpha)> = \sum_{\alpha} f(\alpha)\rho_{\alpha}$ i.3
Part 3: Lattices

Renormalization for d-2 Ising model

\[ Z = \text{Trace}_{\sigma} \exp(W_K(\sigma)) \]

Imagine that each box in the picture has in it a variable called \( \mu_R \), where the \( R \)'s are a set of new lattice sites with nearest neighbor separation 3\( \sigma \). Each new variable is tied to an old one via a normalization matrix

\[ G(\mu, \sigma) = \prod g(\mu_R, \sigma) \]

where \( g \) couples the \( \mu_R \) to the \( \mu \)'s in the corresponding box. We take each \( \mu_R \) to \( e \pm 1 \) and define \( g \) so that

\[ \sum_\mu g(\mu, \sigma) = 1. \]

For example, \( \mu \) might be defined to be an Ising variable with the same sign as the sum of \( \sigma \)'s in its box.

Fewer degrees of freedom produces \textit{“block renor}.

\[ \text{iterations} \qquad \bullet \quad 4 \quad 3 \quad 2 \quad 0 \]

\[ \text{iterations} \qquad \bullet \quad 0 \quad 2 \quad 3 \quad 4 \quad 6 \]

\[ \text{flow} \quad \bullet \]

- stable fixed point
- unstable fixed point

Perimeter Institute Lecture Notes on Statistical Physics: part I: Overview Version 1.7 9/13/09  Leo Kadanoff
Part 4: Random Walks & Diffusion

http://particlezoo.files.wordpress.com/2008/09/randomwalk.png
Part 5: Statistics of Motion

Albert Einstein (1905) explained this dancing by many, many collisions with molecules in fluid

\[ \frac{dp}{dt} = \ldots + \eta(t)p/\tau \]

\[ p = (p_x, p_y, p_z) \quad \eta = (\eta_x, \eta_y, \eta_z) \]

\( \eta(t) \) is a Gaussian random variable resulting from random kicks produced by collisions. Since the kicks have random directions \( \langle \eta(t) \rangle = 0 \). Different collisions are assumed to be statistically independent.

\[ \langle \eta_j(t) \eta_k(s) \rangle = \Gamma \delta(t-s) \delta_{j,k} \]

\[ \partial_t f(p,r,t) + (p/m) \cdot \nabla_r f(p,r,t) - \nabla_r U(r,t) \cdot \nabla_p f(p,r,t) = \text{effects of collisions} \]
particle statistics, i.e. the symmetry properties of the particles’ wave functions, have a major role in determining the behavior of many interesting physical systems. This is especially true when the system is degenerate, i.e. there is a sufficiently high density of identical particles so that there could be a substantial overlap of the wave functions involved. Important degenerate systems include:

for fermions:
• the electrons in atoms

for non-conserved bosons
•

for conserved bosons:
•
Part 7: Phase Transitions and Mean Fields

phases of matter:

- [Image of an iceberg]

which symmetries of nature have been lost in the snowflake?

- [Image of a snowflake]

are they really lost?

[Image of a snowflake]

[Image of an iceberg]

[Image of a snowflake]

http://azahar.files.wordpress.com/2008/12/snowflake_.jpg

http://azahar.files.wordpress.com/2008/12/snowflake_.jpg
Part 8: After Mean Fields: Big Words

Universality:
In appropriate limits, very different systems can have essentially identical properties

Scale Invariance
Systems look the same at different spatial scales

Renormalization
Take advantage of scale invariance and universality to produce a theory of phase transitions.
A start:

Ising system has as its basic variable a spin, $\sigma_z$ which takes on the values $\pm 1$. We shall use the abbreviation, $\sigma$ for this spin.

The behavior of a physical system is described by its Hamiltonian. If we put this spin in a magnetic field in the $z$-direction it has a Hamiltonian $H = -\mu B_z \sigma$.

Statistical Mechanics is defined by a probability. Here the probability is

$$\rho(\sigma) = (1/z) \exp[-H/(k_B T)] = (1/z) \exp[-H/(k_B T)] = (1/z) \exp[\mu B_z \sigma / (k_B T)]$$

We describe this by using the abbreviation, $h$, for the parameters in the probability

$$\rho(\sigma) = (1/z) \exp(h \sigma) \quad h = \mu B_z / (k_B T)$$

normalization: total probability $= 1 = \rho(1) + \rho(-1) = (1/z) \exp(h) + (1/z) \exp(-h)$

therefore $z = \exp(h) + \exp(-h) = 2 \cosh h$

average $X = <X> = \sum_{\alpha} \rho(\alpha) X_\alpha$

therefore $<\sigma> = \rho(1)1 + \rho(-1)(-1) = 1/(2 \cosh h) \{\exp(h) - \exp(-h)\}$

$= (2 \sinh h) / (2 \cosh h) = \tanh h$
Averages from Derivatives

\[ z = \sum_{\sigma} \exp(h\sigma) = 2 \cosh h \]

\[ \frac{d(\ln z)}{dh} = \sum_{\sigma} \sigma \exp(h\sigma) / z = \langle \sigma \rangle = \tanh h \]

\[ \frac{d^2(\ln z)}{(dh)^2} = \sum_{\sigma} (\sigma - \langle \sigma \rangle)^2 \exp(h\sigma) / z = \langle (\sigma - \langle \sigma \rangle)^2 \rangle > = 1 - \langle \sigma \rangle^2 = 1 - (\tanh h)^2 \]

All derivatives of the log of the partition function are thermodynamic functions of some kinds. As I shall say below, we expect simple behavior from the log of \( Z \) but not \( Z \) itself. The derivatives described above are respectively called the magnetization, \( M = \langle \sigma \rangle \) and the magnetic susceptibility, \( \chi = dM/dH \). The analogous first derivative with respect to \( \beta \) is minus the energy. The next derivative with respect to \( \beta \) is proportional to the specific heat, or heat capacity, another traditional thermodynamic quantity. The derivative of partition function with respect to volume is the pressure.

Note how the second derivative gives the mean squared fluctuations

Homework: Read Chapters 1 and 2 in textbook.

Show that \(-d(\ln Z)/d\beta = E = \langle H \rangle\) and \(d^2(\ln Z)/d\beta^2 = \langle (H-\langle H \rangle)^2 \rangle\) and

If \( \ln Z = \text{const} + N \ln \Omega + N \gamma \ln T \), with \( \Omega \) being the volume, find the average pressure and its fluctuations.

Do you know what intensive and extensive mean in statistical physics?