p-adic Strings and Tachyon Condensation

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1 Introduction

Bosonic string theory emerged in the seventies as a method for explaining hadron interactions. As the theory developed, it was discovered that it naturally produced a massless spin 2 particle. Since this had no known place in hadron interactions, and since the correct SU(3) guage theory was being developed the idea was dropped. Eventually, Scherk and Schwarz realized that this spin 2 particle was in fact the graviton! Thus, the theory which had failed as an explanation of hadrons was now being offered as an explanation of everything. However, the original formulation, *bosonic* string theory was dropped in exchange for *supersymmetric* string theory for two reasons, namely, the bosonic theory could not produce the fermions we see in nature, and the theory contained tachyons.

Tachyons are particles with negative mass squared which propagate faster than the speed of light. As we know from quantum field theory, the lagrangian for a spin zero particle such as the tachyon contains a term which goes like $-1/2m^2$ times the field strength squared, and this term acts like the negative of the potential energy. Hence, a particle with negative mass squared will act as if it had a negative potential energy. Hence, increasing the tachyon field strength throughout space will effectively be like lowering the potential energy of the system. Furthermore, as we know from classical mechanics, a dynamical system will tend to evolve in such a way as to minimize it's potential energy. Thus, if a tachyon is present in the theory, any given state in the theory will tend to emit more tachyons. We therefore run into the possible problem of having a never ending cascade of tachyons. This would make the vacuum itself inherently unstable, in which case no physics processes could be reliably calculated.

At first glance, this conclusion seems inevitable, but in fact, it relies upon an incorrect assumption about the nonperturbative structure of string theory. Namely, the hidden idea that tachyons in a tachyon background are still tachyons! To see this, suppose we apply some small perturbation to the vacuum, for example, we add one tachyon per unit volume. These will most likely emit other tachyons since this will increase kinetic and decrease potential energy. Now, once there are a large number of tachyons in the vacuum, any new particles created will be swimming in a sea of tachyons and its effective mass and coupling strengths will be different, much like a body's mass feels different if it's in water. Hence, we cannot conclude that the mass squared is still negative, it may in fact be positive! Thus, bosonic string theory may in fact have a stable vacuum. The questions then emerge, what is this vacuum, what is the process by which this vacuum is reached, and what does physics look like around this vacuum? In short, how do tachyons condense?

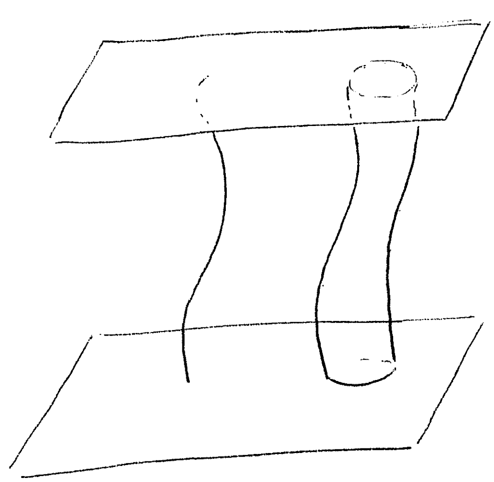
One might wonder why we should bother with worrying about such things when the supersymmetric theory is known to be tachyon free. A quick answer is that even in the supersymmetric theory, we may still have tachyons for certain background solutions. Secondly, and more importantly, it may be possible that bosonic string theory actually teaches us nonperturbative things about supersymmetric string theory. For example, there exists evidence that the supersymmetric theory in fact emerges from the compactification of the $SO(2^{13})$ bosonic string theory on an $E_8 \times SO(16)$ root lattice [1]. It has been conjectured [1] that in fact such a compactification could emerge dynamically through the process of tachyon condensation, in which case supersymmetric string theory could be predicted by bosonic string theory. Although it is far from clear that this is the case, this possibility would strengthen both the

unity of string theory, and the power of the bosonic theory.

Unfortunately, perturbative string theory is incapable of providing us with information about the correct vacuum, much less can it tell us whether or not the root lattice of $E_8 \times SO(16)$ is naturally selected. This is because perturbation theory only works when a small number of particles are created around a given vacuum, not when it is in fact the entire vacuum we wish to calculate! As an added complication, string theory contains an infinite number of different types of particles, all of which are affected by the tachyon field. Hence, information about the true vacuum may be obtained only through indirect means. Despite the difficulties involved, a very concrete picture of open string tachyon condensation has emerged over the past few years, through studies of p-adic string theory and string field theory (SFT). In this paper, I will discuss the applications of p-adic strings to tachyon condensation, and discuss some possibilities for extending p-adic string theory to higher order in perturbation theory.

2 Present Status of Knowledge

A string, being a finite one-dimensional object, may be either open or closed. An open string has two boundaries, and we must therefore place boundary conditions on them to insure that the Polyakov action is extremized when the string propagates according to the equation of motion $\Delta X^{\mu} = 0$. One possible boundary condition is to declare that the ends are constrained to move along some q+1 dimensional hypersurface in 26 dimensional spacetime. The surface upon which the open strings end is a nonperturbative object which inherits a life of its own despite its seemingly auxiliary introduction here. It is called a Dq-brane.



- Pair of Dq-branes sharing an open string and exchanging a closed string.

Now, these Dq-branes may emit and absorb closed strings corresponding to gravitons. Hence, according to classical general relativity, they carry some energy. This applies even when $q = 25$ and the hypersurface is actually all of spacetime. It has been conjectured by A.Sen [2] that the energy of the D25-brane is equal to the height of the tachyon potential and that furthermore, the process of tachyon condensation corresponds to an annhilation of the corresponding brane. When the D-brane is gone, the open string has nowhere to end, and we should be left with a pure closed string theory. To summarize:

Sen's Conjectures

1. The height fo the open string tachyon potential equals the mass of the D25-brane.
2. Lower dimensional Dq-branes are tachyonic solitons.
3. Result of open string tachyon condensation is the closed string theory.

The first two conjectures, as shall be shown, have recieved numerous verifications in both p-adic string theory and SFT. In particular, using a level truncation approximation scheme in Witten's cubic SFT, one can get up to 99.91% agreement for the first conjecture with a tenth order approximation [3], and, for the second conjecture, one may explicitly construct the tachyonic solitons in p-adic string theory which correspond to Dq-branes [4]. The third conjecture is the least well verified. Understanding this withing p-adic string theory will require that we first determine how closed strings arise within the open string sector. This well be addressed in a later section.

3 p-adic String Theory

Numerous connections have arisen between string theory and number theory. To give an example, closed string scattering amplitudes in string theory are calculated as correlators on compact Riemann surfaces modulo a $diff \star Weyl$ symmetry. Such surfaces, as shown by Weyl, are "algebraic varieties", that is, they may be written as the level surface of some complex polyonomial. These structures are in the domain of algebraic geometry which provides a framework for extending the notion of a Riemann surface to continuous fields

other than the complex numbers C. Hence, it seems natural to attempt to formulate string theory using other number systems. Once we demand that this new field have a natural translation invariant measure and that scattering amplitudes be expressed in a way analogous to the usual, or "Archimedean", case, we are left with the p-adic numbers and their extensions [4].

Extending string theory to the p-adic numbers is perhaps not as radical as it may at first seem. After all, spacetime is still a real manifold and quantum amplitudes are still valued in the complex numbers. Furthermore, whenever we perform an experiment, we are fundamentally limited by the accuracy of our measuring equipment. Hence, our answers must be expressed as a number with some finite number of decimal places, plus or minus some error. Thus, all that we need to consistently do physics is the rationals. As we shall see, the p-adic numbers contain the rationals and so are suitable for physics. Finally, no one has ever observed a quantum string of any sort so it is counterproductive to *a priori* place restrictions on such a strings possible nature. Now that we are suitably motivated, let us proceed to construct p-adic strings. In the following, let Q be the rationals, P the prime integers, Z the integers, and C the complex numbers.

Definition 1 *For any $p \in P$, and $\forall x \in Q$, we may uniquely write $x = \left(\frac{a}{b}\right) \cdot p^n$ with $a, b, n \in Z, b > 0$ and p, a, b pairwise mutually prime. Define the p-adic norm of x to be $|x|_p = p^{-n}$, and define Q_p to be the completion of Q wtih respect to the norm $|\cdot|_p$. Also, let Z_p be the subset of Q_p with p-adic norm less than or equal to one.*

Thus, for all p, Q_p naturally contains Q but with a different norm. The set Q_p was constructed by adding the minimal amount of points necessary to guarantee that Cauchy sequences converge. The sequence $1, p, p^2, p^3, \ldots, p^n$ has decreasing p-adic norm, just like the base 10 expansion $1, 10^{-1}, 10^{-2} \ldots$ in the real case. By an analgous proof to the real case, one may show that any $x \in Q_p$ may be written as an infinite sum $x = \sum_{k \geq n} a_k \cdot p^k$, where $a_k < p$ and $n \in Z$. The sequence obviously converges and the fact that any x may be written as such a sequence follows since if we try and minimize $|x - \sum_{k \geq n} a_k \cdot p^k|_p$ with respect to the a_k, we see that it must be smaller than p^{-k} for all k.

Just as the boundary of the usual Archimedean open string world sheet is R, we will take the boundary of the p-adic open string to be Q_p. Scattering amplitudes may be obtained by taking Archimedean amplitudes and replacing the integrals over R by integrals over Q_p and the real norm by the p-adic norm [3]. For example the n-point open string tree level tachyon scattering amplitude becomes:

$$A(\{k_i\}_{1\leq i\leq n}) = \int_{R^{n-3}} dx_2 dx_3 \dots dx_{n-2} \prod_{2\leq i\leq n-2} |x_i|^{k_1\cdot k_i} |1-x_i|^{k_{n-1}\cdot k_i} \prod_{2\leq i<j\leq n-2} |x_i - x_j|^{k_i\cdot k_j} \tag{1}$$

$$\Rightarrow$$

$$A(\{k_i\}_{1\leq i\leq n}) = \int_{Q_p^{n-3}} dx_2 dx_3 \dots dx_{n-2} \prod_{2\leq i\leq n-2} |x_i|_p^{k_1\cdot k_i} |1-x_i|_p^{k_{n-1}\cdot k_i} \prod_{2\leq i<j\leq n-2} |x_i - x_j|_p^{k_i\cdot k_j} \tag{2}$$

It is a non-trivial check that these p-adic amplitudes actually retain all the desirable features of the original amplitudes such as Lorentz invariance, crossing symmetry, factorization, and the absence of simultaneous poles in incompatible channels. Therefore they represent a consistent set of S-matrix amplitudes. The primary differences from the Archimedean case are the triviality of the spectrum (just one tachyon) and the appearence of an infinite collection of complex poles representing a breakdown of locality at the Planck scale. Thus we have a method for calculating p-adic amplitudes given Archimedean amplitudes, but it does not really tell us where the p-adic amplitudes come from. To answer this requires a formulation of p-adic string theory based upon a world sheet approach in which scattering amplitudes are computed as correlators of operators on some type of p-adic Riemann surface. This problem was solved in reference [6] in which the Bruhat-Tits, B_p, tree was identified as the proper p-adic world sheet.

Definition 1 *Define the Bruhat-Tits tree to be $B_p = PGL(2, Q_p)/PGL(2, Z_p)$, where, for any ring K, PGL(2,K) is the group of fractional linear transformations of the projective line $P^1(K)$. The action of $PGL(2, Q_p)$ on $P^1(K)$ is defined as follows: For $\mathbf{X} \in PGL(2, K)$ written,*

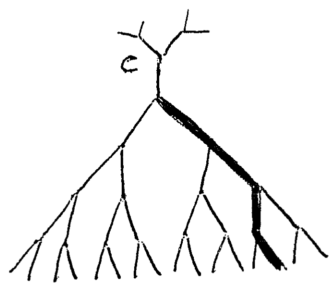
$$\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

we define the action on $k \in PGL(2, K)$ as:

$$X : z \rightarrow \frac{az+b}{cz+d} \tag{4}$$

This has a natural analogue of the interior of the usual string world sheet, which may be written as $H = SL(2,R)/SO(2)$, where H is the upper half plane. The relationship between $SO(2)$ and $PGL(2,Z_p)$ is that they are both the maximal compact subgroup of their respective groups, which makes the two spaces naturally analogous. Furthermore, B_p is endowed with a natural $GL(2,Q_p)$ invariant metric, labeled d(x,y), which gives it a discrete topology. We may visualize B_p as an infinite tree with p+1 branches at each vertex and no loops. In the tree language, d(x,y) is simply the length of the unique path from x to y. The action by $GL(2,Q_p)$ is determined once we say where one vertex and its p+1 legs go.

To identify the boundary with Q_p, designate some vertex as the "center" C, and then choose one branch which shall correspond to Z_p. Given $x \in Z_p$, we may write it as $x = \sum_{k \geq 0} a_k \cdot p^k$ where $0 \leq a_k < p$. Then, form the ray from C to the boundary by moving downwards, and at the i^{th} step, taking the a_i^{th} branch. For example, when $p = 2$, and $x = 1 + 1 \cdot p + 0 \cdot p^2 + 1 \cdot p^3$, we have the following path:



Thus, we may identify Z_p with one of the branches from C. Similarly, we may take the remaining branches and identify them with $Q_p - Z_p = \{x \in Q_p : |x|_p > 1\}$ by simply taking the inverse of the map just defined. If we attach these branches, we obtain a map from the whole boundary of B_p to Q_p, or rather, to $P^1(Q_p)$ since in inverting the region containing zero we pick up a point at infinity, giving us $P^1(Q_p)$.

Given a vertex, v, which is a distance R from C, one may define the measure of the branch B_v extending from this vertex to be $\mu(B_v) = p^{-R+1}$. this defines an integration measure dμ ont the boundary of B_p. There is also a canonical integration measure, dx, on Q_p defined by translation invariance. The measures are related by :

$$d\mu = \begin{cases} dx & \text{for } |x|_p \leq 1 \\ dx \cdot |x|_p^{-2} & \text{for } |x|_p > 1 \end{cases} \tag{5}$$

-where the $|x|_p^{-2}$ is the Jacobian induced from inverting the branch mapped to $Q_p - Z_p$.

To complete the construction, we need the analogue of the Polyakov action on the world sheet. Let X^μ denote the embedding of B_p into spacetime. A notion of a p-adic Laplacian Δ_p may be obtained by requiring it to be second order and invariant under $GL(2, Q_p)$ transformations. Using this Laplacian, the Polyakov action has the natural analogue:

$$S_p[X^\mu] = -1/2 \cdot C_p \cdot \sum_{z \in B_p} X^\mu(z) \cdot \Delta_p X_\mu(z) \tag{6}$$

-where C_p is the p-adic tension, which is related to the Regge slope by $C_p = \frac{1}{2\pi\alpha'}$ and Δ_p is the p-adic version of $\Delta = \partial^2$, defined by:

$$\Delta_p \Phi(z) = \sum_{d(z,w)=1} \Phi(w) - (p+1) \cdot \Phi(z) \tag{7}$$

Now, we may define scattering amplitudes for open string vertex operators using the path integral, just as is done for the usual Archimedean case. So, given open string vertex operators $V_1(z_1), V_2(z_2) \ldots V_n(z_n)$, inserted at points $z_1, z_2, \cdots z_n \in E_R = \{z \in B_p : d(C, z) = R\}$, and letting Z be the vacuum to vacuum amplitude, define:

$$\langle V_1 \cdots V_n \rangle = (Z^{-1}) \cdot \lim_{R \to \infty} \sum_{\{z_i\}_{i \leq n} \in E_R} \int DX e^{-S_p[X^\mu]} \cdot V_1(z_1) \cdots V_n(z_n) \tag{8}$$

In words, we insert the vertex operators at points z_i a distance R from C, do the path integral, sum over z_i, and then take the limit as R approaches infinity. We end up with a path integral with insertions on the boundary of the tree, in analogy with the Archimedean theory in which open strings are integrated over the boundary of the world sheet.

Example [6]: Consider the case where the vertex operators are tachyon operators $V_n = e^{i \cdot k_n^\mu \cdot X_\mu(z_n)}$, where k_n is a momentum. We then have:

$$\langle V_1 \cdots V_n \rangle = (Z^{-1}) \cdot \lim_{R \to \infty} \sum_{\{z_i\}_{i \leq n} \in E_R} \int DX e^{-S_p[X^\mu] + i \sum_{1 \leq i \leq n} k_i \cdot X(z_i)} \tag{9}$$

This is evaluated by writing $X(z)$ as the sum of a classical solution, X_{cl} and a fluctuation piece, X_{fl}, which we integrate over (for a finite dimensional integral over space, this just corresponds to redefining the zero point). Define:

$$X^\mu(z) = X^\mu_{cl}(z) + X^\mu_{fl}(z) \tag{10}$$

-where X_{cl} obeys the equation of motion obtained from varying the action. If we let δ denote the Kronecker delta, the equation of motion is:

$$C_p \Delta_p X^\mu_{cl}(z) = -i \sum_{1 \leq i \leq n} k_i \cdot \delta_{z,z_i} \tag{11}$$

If we plug (10) into (8) and use the equation of motion (11), then we see that the path integral factorizes into an integral over the fluctuation piece, times a classical piece:

$$= (Z^{-1}) \cdot (\int DX_{fl} e^{-S_p[X_{fl}]}) \cdot (\lim_{R \to \infty} \sum_{\{z_i\} \in E_R} e^{\frac{i}{2} \sum_{1 \leq i \leq n} k_i \cdot X_{cl}(z_i)}) \tag{12}$$

The integral over X_{fl} cancels the partition sum Z. All that remains is to determine X_{cl}. This may be written in terms of the Neumann propagator $N(z,w)$ which satisfies:

$$N(z,w) := \frac{1}{2}(d(z,w) - d(w,C) - d(z,C)) \tag{13}$$

$$\Delta_p N(z,w) = \delta_{z,w} - \delta_{z,C} \tag{14}$$

N(z,w) may also be described as follows, given z and w, draw the unique path from each of them to C. The negative of the length of the overlap between these two paths is N(z,w). The result, [7], is independent of the choice of C if momentum is conserved (more on this later). The classical solution may be written:

$$X^\mu_{cl} = -iC_p^{-1} \sum_{1 \leq i \leq n} k^\mu_i N(z, z_i) \tag{15}$$

If we substitute this into our formula for the amplitude (12) then we get:

$$= \lim_{R \to \infty} \sum_{\{z_i\}_{1 \leq i \leq n} \in E_R} e^{(2C_p)^{-1} \sum_{1 \leq i,j \leq n} k_i \cdot k_j N(z_i, z_j)} \tag{16}$$

The sum in the exponent of (16) may clearly be written as:

$$\sum_{1 \leq i,j \leq n} k_i \cdot k_j N(z_i, z_j) = - \sum_{1 \leq i \leq n} k_i^2 \cdot d(C, z_i) + 2 \sum_{1 \leq i < j \leq n} k_i \cdot k_j N(z_i, z_j) \tag{17}$$

Furthermore, in the $R \to \infty$ limit, the sum over $\{z_i\} \in E_R$ may be replaced by an integral with respect to the previously defined measure $d\mu$ via:

$$\sum_{z \in E_R} \to p^R \int_{\partial B_p} d\mu \tag{18}$$

Making these alterations in (16) we arrive at:

$$= \lim_{R \to \infty} p^{nR} e^{-(2C_p)^{-1} k_i^2 nR} \cdot \Big(\int_{(\partial B_p)^n} \prod_{1 \le j \le n} d\mu(z_j) e^{C_p^{-1} \sum_{i<j} k_i \cdot k_j N(z_i, z_j)} \Big) \tag{19}$$

In order that the $R \to \infty$ limit be non trivial, we need the two R dependent terms to cancel each other, which determines the mass shell condition:

$$k_i^2 = 2C_p ln(p) \tag{20}$$

Since C_p is some dimensional parameter, we may arbitrarily choose it so that the mass shell condition agrees with the Archimedean theory. Thus we choose [1]:

$$C_p = ln(p)^{-1} \tag{21}$$

$$k_i^2 = 2 = -m^2 \tag{22}$$

With this choice, the answer becomes:

$$= \int_{(\partial B_p)^n} \prod_{1 \le j \le n} d\mu(z_j) p^{\sum_{i<j} k_i \cdot k_j N(z_i, z_j)} \tag{23}$$

If we write this in terms of the p-adic measure dx rather than the measure $d\mu$, and we use $PSL(2, Q_p)$ symmetry to fix three vertex operators, then this becomes:

$$= \int_{(Q_p)^n} \prod_{2 \le i \le n-2} (dx_i) |x_i|_p^{k_1 \cdot k_i} |1 - x_i|_p^{k_{n-1} \cdot k_i} \prod_{2 \le j < k \le n-2} |x_j - x_k|_p^{k_j \cdot k_k} \tag{24}$$

This is identical to formula (2), in which we simply replaced real numbers and norms by their p-adic counterparts. Hence the Bruhat-Tits tree appears to be the true p-adic counterpart of the world sheet. It is a remarkable fact about this formula, that for n=4, the Archimedean amplitude is built out

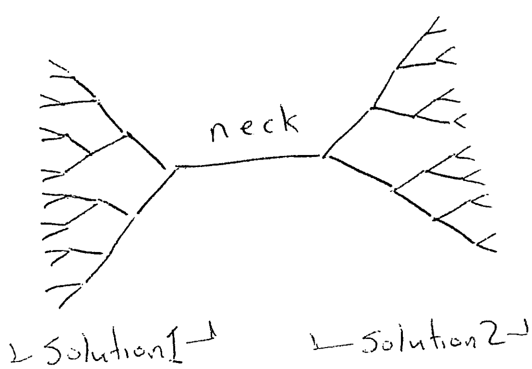
[1] Note that our tension and slope depend on p since we chose to use e as the base of our exponent. If we had used p as the base, then this dependence would have disappeared.

of the p-adic formula in the following sense. If we let $A(\{k_i\}_{i\leq 4})$ be the Archimedean amplitude and $A_p(\{k_i\}_{i\leq 4})$ be the p-adic amplitude, then [4]:

$$A(\{k_i\}) \cdot \prod_p A_p(\{k_i\}) = 1 \tag{25}$$

The above formula is an example of an adelic product formula which incorporates all the p-adic fields and the reals on equal footing. This relationship is one of the strongest pieces of evidence that the approach to p-adic strings outlined above is correct. Despite this success, there is as of now no known generalization for n larger than 4.

Now that we have the correlator for tachyons, the question immediately arises, what are the other possible vertex operators V_i? This amounts to asking what are the conserved quantum numbers, since we cannot construct an operator representing a hypothetical state if nothing about that state is preserved during interactions. Thus we are led to ask, what states may propagate into other states via a solution of the classical equations of motion. To answer this, suppose we have two solutions to the classical equations of motion, one on the branch leading to Z_p and the other on the branches leading to $Q_p - Z_p$.



We want to know if these solutions may be continued to the whole tree. Now, obviously, this is trivial since we need only check consistency at the two points z and w between Z_p and $Q_p - Z_p$. Both solutions uniquely determine the difference in the value of the X^μ coordinate at these two points, and this is unchangeable. Since we may add a zero momentum solution to either half to make one of the values match up, the only real consistency requirement is that the value of the difference between these two points be the

same. In other words, the solutions are parametrized by a set of numbers, one for each spatial coordinate, and we know these must be the components of the momentum. To state this more simply, all the information about the state is lost as the solution is forced to squeeze across the neck, except the momentum. In more generality, suppose we had some solution on a neighborhood of the boundary, say for all z such that $d(C, z) > R$. The following says that we may do the same thing as above, add some constants to make sure the solutions match up in the middle and then just require momentum conservation.

Theorem 1 *Suppose we are given a Bruhat-Tits tree with center C and some solution $X(z)$ to the classical equations of motion is given for $d(z, C) > R$. Let $\sum k$ be the net momentum flux into the tree, given by,*

$$\sum k = \sum_{1 \leq i \leq p+1} \Big(\sum_{d(z,C)=s, z \in B_i} X(z) - \sum_{d(z,C)=s+1, z \in B_i} X(z) \Big) \cdot p^{-s} \qquad (26)$$

where B_i labels a branch protruding from C and the convention is that C is in all of the branches. Then $\sum k$ so defined is independent of s and the solution X(z) extends to the whole tree iff $\sum k = 0$.

Proof: That $\sum k$ is independent of s may be checked by comparing the value of k at s and s+1. Using the equation of motion, one sees that the two values are equal as required.

Now, the solution on the boundary should uniquely determine a solution on the interior. We have a problem, however, since each vertex at level s+1 uniquely determines the vertex above it at level s if we try to satisfy the equations of motion, and each vertex at level s determines the one above it, all the way up to level 1. It is possible that the value of X(z) at level s determined in this way is different for different branches below it at level s+1. Thus we must shift each of the solutions at level s+1 by some trivial zero momentum solution to guarantee that they all give consistent values for X(z) at level s. We know that we may do this since there are p^{s+1} branches at level s+1, giving us that many degrees of freedom, and there are $\sum_{1 \leq j \leq s} p^j = \frac{p^{s+1}-p}{p-1}$ vertices above those branches giving us that many constraints. Thus we have more freedom than constraints so that we may safely shift the solution on the boundary to guarantee consistency up to level 1.

Once we have found a solution up to level 1, the question remains if this can be extended to the center C, which is at level 0. One may easily see that this is precisely the requirement that k be zero when $s = 0$, or equivalently,

that k be zero for all s. Thus, momentum conservation is the only real requirement in forming a solution.

Since k is the only conseved quantity, the only possible vertex operator is $e^{ik\cdot X}$, and the only particles are tachyons. One may attempt to include gauge particles using Chan-Patton factors, however, this results only in a theory with global, rather than local gauge symmetry [8].

4 Applications of p-adic Strings to Tachyon Condensation

Due to the fact that there are only tachyons in the spectrum, p-adic string theory is an ideal setting for studying tachyon dynamics. One may, in fact, explicitly solve for all n-point tachyon scattering amplitudes and thus reconstruct the effective tachyonic lagrangian. Letting θ be the open string tachyon field, the lagrangian is [4]:

$$L_p = \frac{p}{p-1} \cdot (\frac{1}{2}\theta p^{-\frac{\partial^2}{2}-1}\theta + g^{-2} \cdot \frac{p}{p-1}(1 + \frac{g\theta}{p})^{p+1} - \frac{\theta}{g} - g^{-2}\frac{p}{p-1}) \quad (27)$$

Note that the tachyonic potential has a minimum at $\theta = -\frac{p}{g}$. This provides an explicit example of a purely tachyonic theory which nevertheless has a stable vacuum. The process of condensation may be simply achieved by defining $\phi = 1 + \frac{g\theta}{p}$, which is just the tachyon field expanded around it's ground state. In terms of ϕ, the lagrangian becomes:

$$L_p = g^{-2}\frac{p^2}{p-1}(-\frac{1}{2}\phi p^{-\frac{\partial^2}{2}}\phi + \frac{\phi^{p+1}}{p+1} + \frac{p-1}{2(p+1)}) \quad (28)$$

The equation of motion one obtains is:

$$p^{-\partial^2/2}\phi = \phi^p \quad (29)$$

In d-dimensional spacetime this is equivalent to (after a Wick rotation):

$$\phi(x) = (2\pi ln(p))^{-d/2}\int d^d y e^{-\frac{(x-y)^2}{2\cdot ln(p)}}\phi^p \quad (30)$$

This admits [4] factorized solutions of the form $\phi(x_1, \cdots, x_d) = f_1(x_1)\cdot \cdots \cdot f_{d-q-1}(x_{d-q-1})$ for $q \leq d-1$, x_d representing the time direction, and:

$$f_i(x) = A \cdot e^{-\frac{x^2}{a^2}} \quad (31)$$

where

$$A = -p^{\frac{1}{2(p-1)}} \tag{32}$$

$$a^2 = \frac{p \cdot ln(p)}{p-1} = 2\pi\alpha_p' \cdot \frac{p}{p-1} \tag{33}$$

This represents a solution which is constant in q spatial directions and whose energy density is localized to within a Planck's length of the q dimensional hyperplane $x_1 = x_2 = \cdots = x_{d-q-1} = 0$. In the classical limit, $\alpha_p' \to 0$, we may find solutions corresponding to an arbitrary number of parallel hyperplanes at various positions. Gloshal and Sen have shown that, in fact, these correspond to Dq-branes [5]. In particular, if we let T_q be the tension of a Dq-brane, obtained by integrating the lagrangian minus the zero point energy for this solution, then we have:

$$T_q = \int dx^{d-q-1}(L_p - \frac{p^2}{2g^2(p+1)}) \tag{34}$$

$$T_q = \frac{p^2}{2g^2(p+1)} \cdot (2\pi p^{\frac{2p}{2p-1}} \frac{ln(p)}{p^2-1})^{\frac{d-1-q}{2}} \tag{35}$$

This leads to:

$$\frac{T_q}{T_{q-1}} = \frac{(p^2-1)^{\frac{1}{2}}}{p^{\frac{p}{p-1}}} \cdot (2\pi(\alpha_p')^{\frac{1}{2}})^{-1} \tag{36}$$

where again α_p' is the Regge slope which is related to the tension C_p in the usual way, $\frac{1}{2\pi\alpha_p'} = C_p$, so $\alpha_p' = \frac{ln(p)}{2\pi}$ with our previous choice of $C_p = (ln(p))^{-1}$ as in (21). Thus, we see that the ratio of tensions is independent of q and approaches the Archimedean value in the $p \to \infty$ limit. Furthermore, it was shown that the world volume action of the Dq-brane is equivalent to the Archimedean world volume action. Thus, p-adic Dq-branes may be explicitly realized as solitons, confirming Sen's second conjecture. Also, if we plug in $q = d-1$ into the formula for T_q, it is easy to verify that the height of the tachyon potential, $\frac{p^2}{p+1}$, is precisely the T_{d-1} brane tension, verifying Sen's first conjecture. As for the third, namely that the open string vacuum is the noncondensed closed string vacuum, we cannot verify this until we understand closed p-adic strings.

5 Closed p-adic Strings

The world sheet of bosonic string theory can naturally be thought of as a complex manifold, as was mentioned in the introduction. Now, in Archimedean

string theory, closed string operators are integrated over this complex world sheet while open string operators are integrated along the boundary. The field of complex numbers, C is obtained from R by extending by one element, i, that satisfies some quadratic equation, namely $x^2+1=0$. Hence C is called a "quadratic extension" of R. Therefore, the natural p-adic analogue of C seems to be a quadratic extension of Q_p, denoted by $Q_p(\tau)$. Thus one might expect that in formulating a closed p-adic string theory we should integrate operators over $Q_p(\tau)$. Unlike the Archimedean case where there is just one quadratic extension, the p-adics have three inequivalent extensions obtained by choosing τ to satisfy one of $\tau^2-\varepsilon=0$, $\tau^2-p=0$, or $\tau^2-\varepsilon p=0$, where ε is a $(p-1)^{th}$ root of unity. The last two extensions produce identical four point functions while the first one is different. Together they give an infinite number of adelic product formulae with the Archimedean amplitude. Given this situation, it is unclear how to define a unique closed theory, even at tree level. Ignoring these issues for the moment, we will consider the case where $\tau=\varepsilon^{\frac{1}{2}}$, since this has been studied in the most detail. One is led to the following lagrangians by computing the tree level interactions, where θ is the open string field before tachyon condensation, and $\phi=1+\frac{g\theta}{p}$ is the open string field expanded about the minimum of the tachyon potential, and similiarly, ρ is the closed string field before condensation, and $\psi=1+\frac{h\rho}{p^2}$ is the field expanded about it's minimum [4]:

Before Condensing

$$\begin{aligned}L_p = &-\frac{p}{2(p-1)}\cdot\theta p^{-\partial^2/2-1}\theta-\frac{p\theta}{g(p-1)}-\frac{p^2}{2(p^2-1)}\cdot\rho p^{-\frac{\partial^2}{4}-2}\rho \\ &+\frac{p^2}{g^2(p^2-1)}\cdot(1+\frac{g\theta}{p})^{p+1}\cdot(1+\frac{h\rho}{p^2})^{\frac{p(p-1)}{2}}-\frac{p^2\rho}{h(p^2-1)} \\ &+\frac{p^2}{g^2(p^2-1)}\cdot(1+\frac{h\rho}{p^2})^{\frac{p(p-1)}{2}}+\frac{p^4}{h^2(p^4-1)}\cdot(1+\frac{h\rho}{p^2})^{p^2+1} \\ &-\frac{p^4}{h^2(p^4-1)}\end{aligned} \tag{37}$$

After Condensing

$$\begin{aligned}L_p = &-\frac{p^2}{2g^2(p-1)}\cdot\phi p^{-\partial^2/2}\phi-\frac{p^4}{2h^2(p^2-1)}\psi p^{-\partial^2/4}\psi+\frac{p^4}{h^2(p^4-1)}\cdot\psi^{p^2+1} \\ &-\frac{p^2}{g^2(p^2-1)}\cdot\psi^{p(p-1)/2}+\frac{p^2}{g^2(p^2-1)}\cdot\phi^{p+1}\psi^{p(p-1)/2}+\frac{p^2}{2g^2(p-1)} \\ &+\frac{p^4}{2h^2(p^2+1)}\end{aligned} \tag{38}$$

The pure closed string lagrangian may be obtained by setting the open string parameter $1/g$ equal to zero in (38). The equations of motion derived from the above lagrangian (38) are:

$$p^{-\partial^2/2}\phi = \phi^p \psi^{p(p-1)/2} \tag{39}$$

$$p^{-\partial^2/4}\psi = \psi^{p^2} + \frac{h^2(p-1)}{2pg^2} \cdot (\phi^{p+1} - 1)\psi^{p(p-1)/2-1} \tag{40}$$

These are equivalent to:

$$\phi(x) = (2\pi ln(p))^{-d/2} \int d^d y e^{-\frac{(x-y)^2}{2ln(p)}} \phi^p \psi^{p(p-1)/2} \tag{41}$$

$$\psi(x) = (\pi ln(p))^{-d/2} \int d^d y e^{\frac{(x-y)^2}{ln(p)}} (\psi^{p^2} + \frac{h^2(p-1)}{2pg^2} \cdot (\phi^{p+1} - 1)\psi^{p(p-1)/2-1}) \tag{42}$$

It may be shown that the only two solutions are the trivial ones $\psi = \phi = 0$ and $\psi = \phi = 1$. This creates a problem when trying to compare this to the Archimedean case, since there we have Dq-brane solutions in the theory with both open and closed strings. The problem may be due to the fact that the explicit introduction of closed strings in this manner might lead to double counting of the moduli space, since pure open string one loop amplitudes will also give rise to closed strings. Thus, following Zwiebach [9], if we are to introduce a closed string interaction, then we must exclude regions of the moduli space in one loop calculations that are already covered at tree level. Similarly, at the n^{th} order in perturbation theory, we must exclude integrations over regions of the moduli space that were covered at any lower order. To avoid this problem, we must learn how closed p-adic strings emerge at the one loop level of open p-adic strings. This question will be adressed in the next section.

In the Archimedean theory, $g^2 \simeq h < 1$, so we may consider the limit $h << g$ in which case the second term on the right hand side of (42) drops out. We are left with:

$$\psi(x) = (\pi ln(p))^{-d/2} \int d^d y e^{\frac{(x-y)^2}{ln(p)}} \psi^{p^2} x \tag{43}$$

This is, incidentally, also the equation of motion one obtains from a pure closed string theory, ie, by integrating vertex operators over $Q_p(\varepsilon^{1/2})$ with no boundary. This also admits solutions corresponding to Dq-branes which may be written in the factorized form:

$$\psi(x_1, \cdots, x_d) = g_1(x_1) \cdot \cdots \cdot g_{d-q-1}(x_{d-q-1}) \tag{44}$$

$$g_i(x) = Ae^{-x^2/a^2} \tag{45}$$

$$A = p^{\frac{1}{p^2-1}} \qquad a^2 = \frac{p^2 ln(p)}{p^2 - 1} \tag{46}$$

We wish to find the ratio of tensions in terms of the Regge slope. To do this, we must repeat the procedure leading up to (19) and then require that the $R \to \infty$ limit be well defined. The only difference is that, whereas open strings are built on a p+1 branching lattice, closed strings based on the $\varepsilon^{1/2}$ are built on a p^2+1 branching lattice [6]. The extra two in the exponent will shift our earlier result by a factor of two. If we represent closed parameters with an overbar, then our previous calculation now gives:

$$k^2 = 4\overline{C'_p} ln(p) \qquad \overline{C'_p} = \frac{1}{2\pi\overline{\alpha'_p}} \tag{47}$$

One knows from the Archimedean theory that the closed string tension is twice that of the open string. Referring back to (21) this determines:

$$k^2 = 8 \iff m^2 = -8 \qquad \text{and} \quad \overline{\alpha'_p} = \frac{ln(p)}{4\pi} \tag{48}$$

This is exactly the same mass squared of the closed string tachyon as in the Archimedean theory. Hence, the ratio of closed to open string tachyon mass is determined correctly and independently of the requirement of adelization. If we now proceed to calculate the ratio of the tensions, in complete analogy with (36), using (48), we get:

$$T_q = \frac{p^4}{2h^2(p^2+1)} \cdot (p^{\frac{2p^2}{p^2-1}}(p^4-1)^{-1/2} \cdot (2\pi\overline{\alpha'_p}^{1/2}))^{d-q-1} \tag{49}$$

$\Rightarrow$

$$\frac{T_q}{T_{q-1}} = \frac{(p^4-1)^{1/2}}{p^{\frac{p^2}{p^2-1}}} \cdot (2\pi\overline{\alpha'_p}^{1/2})^{-1} \tag{50}$$

This is very similar to the previously calculated ratio (36). In particular, the ratio is independent of q as required for D-branes. A study of the world volume action of these tachyonic solitons in analogy with the analysis done in [5] for the open string case bears out the conjecture that these indeed correspond to Dq-branes.

This presents a problem, common to each of the three quadratic extensions, when we try to interpret the meaning of these solutions in comparison to the Archimedean case. This is because in the Archimedean theory, there is only one type of Dq-brane, which is where open strings are allowed to end. In particular, the tension always goes like $1/g^2$ which is necessary for a consistent coupling to Ramond-Ramond fields, never like $1/h^2 \sim 1/g^4$. In this case, however, it appears that we have two types of Dq-branes. One may check that these new closed string Dq-branes play an exactly analagous role in the condensation of closed string tachyons as the open string Dq-branes played for open string tachyons. For example, comparing the tension of the D25-brane in (49) to the height of the tachyon potential in (38) (set $1/g = 0$ and $\psi = 0$) one sees that they are equal. One might therefore conjecture that these closed strings are more naturally thought of as different types of open strings (p48 [4]). The only way to adress these questions is to determine how closed strings emerge as loops in the open string sector, for then we will understand better the role of $Q_p(\tau)$ from the open string viewpoint. This will be briefly discussed in the next section.

Another problem with interpreting ψ as a closed string field is that, unlike C, $Q_p(\tau)$ is not algebraically closed. Hence, by exactly the same reasoning which allowed us to introduce closed p-adic strings, we may introduce "even more closed strings" [4] which are vertex operators integrated over higher algebraic extensions of Q_p, and then we may introduce "even more even more closed strings" and so on. Now, in Archimedean string theory, it is not an option to introduce closed strings or not - we must do so to preserve unitarity. However, in the p-adic case, it is not now understood how closed strings must "necessarily" be introduced into the open string theory, and indeed, if it were necessary, it would seem that we must introduce the infinite tower of even more closed strings as well. Relating this to the Archimedean case seems difficult.

For the reasons mentioned above, we cannot seek for evidence of Sen's third conjecture within the present framework of p-adic strings. A naive comparison will show that the pure closed string theory and the theory obtained from the full "open and closed" theory are not equivalent. The pure closed theory is:

$$p^{-\partial^2/4}\psi = \psi^{p^2} \tag{51}$$

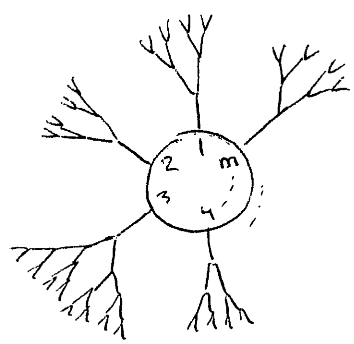
The theory obtained from the "open closed theory by setting $\phi = 0$ in (42), is:

$$p^{-\partial^2/4}\psi = \psi^{p^2} - \frac{h^2(p-1)}{2pg^2} \cdot \psi^{p(p-1)/2-1} \tag{52}$$

These are different, in contradiction with Sen's third conjecture. However, as was already mentioned, there are very likely moduli miscounting problems with the tree level lagrangian. Hence we need to understand higher order amplitudes and the emergence of closed strings from open ones in order to verify Sens's third conjecture in the context of p-adic string theory. This goal has merits independent of tachyons, since, hopefully, it will give us an even better model of other aspects of Archimedean string theory as well. The next section takes a few steps in this direction.

6 Higher Order Amplitudes

The first attempts at loop amplitudes were taken in [7]. There, p-adic world sheets were constructed from the usual Bruhat-Tits tree by making identifications according to the action of some discrete symmetry group, known as a *Schottky* group. This is in complete analogy with the corresponding construction in Archimedean string theory. When the Schottky group is generated by translation along some line in the tree by an amount m, the corresponding world sheet is a loop with m vertices from which protrude $p+1$ branching trees.



The important thing to note, is that this construction gives a surface with only one boundary. Thus, this does not allow us to do a calculation in which two open strings combine to form a closed string intermediate state, which then breaks apart again into two open strings. This diagram, in the Archimedean case, would require that we consider two open string insertions on each of the two boundaries of a cylinder, since this is the surface which propating intermediate closed string sweeps out.

It is important to consider this diagram in the p-adic case, since, for elementary reasons, the pole in this amplitude occurs when the center of mass energy of the incoming open strings is equal to the mass squared of the closed string. Since we already know that the mass of the closed p-adic string is minus eight, this diagram can be used as a check of a given method for calculating loop effects. Furthermore, we wish to be able to determine how closed strings arise "naturally" in the open string sector, for only then may we remove the miscounting problem mentioned earlier and go to higher order in perturbation theory.

Unfortunately, the problem as of now has not been solved. A number of methods for modeling a p-adic surface with two boundaries has been tried [2], all of which produce the wrong mass for the closed string. It thus remains a mystery how to go to higher order in perturbation theory within p-adic string theory. Whatever the final form of the loop expansion may be, it not only must correctly predict the closed string tachyon mass as the pole in the one loop correction to the open string four point function, but, more stringently, it must give us the complete closed string theory by factorizing the open string loop amplitudes. It would be an added bonus if the correct construction could be accompanied by diagrams based on either a $p+1$ or a p^2+1 lattice. Perhaps the lattices of higher order extensions of Q_p must come into play here.

[2]Specifically, inserting extra copies of Q_p into the interior of the tree gives $m^2 = -1/2$ and partitioning the branches of a 1-loop diagram into two boundaries gives an infinite collection of poles.

7 Conclusion

It has been shown that p-adic strings offer an excellent model for tachyon condensation in the open string sector. In the closed string sector, however, p-adic string theory naively predicts D-branes and, to all appearences, the closed string theory is completely analgous to the open string theory. We know however that this is not the case in the Archimedean theory, so the question then emerges how to correctly incorporate closed p-adic strings. To answer this, we must go to higher orders in the open p-adic string perturbation theory. To this end, a number of possibilities for loop diagrams have been eleminated, while the correct theory still remains elusive.

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