January 27, 2010

1 1. Bifurcation Diagram for the Logistic Map

Bifurcation diagrams display how the behavior of a dynamical system changes with a dynamical parameter. The canonical example of an "interesting" bifurcation diagram is the logistic system which is defined as follows: Take the so called logistic map, $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = rx(1 - x)$$

and recursively define $x_{n+1} = f(x_n)$. Note that in order for f to take the unit interval to itself, we must restrict r to be than $0 \leq r \leq 4$. As the dynamical parameter r is increased, the stable equilibria change and branch into n-cycles. n increases with r so that members of various n-cycle becomes similar to the uncertainty in initial x value at which point predictability is list. This can be seen in the logistic map bifurcation diagram seen in Figure 1.

The bifurcation diagram applet takes the following inputs:

1. rStart: The starting value of the dynamical parameter r.
2. rEnd: The final value of the dynamical parameter r.
3. numR: The number of r's that will be used.
4. minX: The minimum starting value used for x.
5. maxX: The maximum starting value used for x.
6. numIters: The number of iterations to be done before data will be taken.
7. numStarts: The number of starting values that will be used.
8. numPoints: The number of points that will be plotted per value of r.

The program uses a while-loop to go through the number the starting value of r and final value of r in intervals of $\frac{\text{rEnd}-\text{rStart}}{\text{numR}}$. At each value of r, a for-loop runs through the desired starting points starting with minX and ending with maxX in intervals of

$\frac{\text{maxX}-\text{minX}}{\text{numStarts}}$. Multiple starting points are useful if more than one stable fixed point exists. At each starting point, the value

$$f^{(\text{numIters})}(x) \tag{1}$$

is computed using another for loop which simply iterates the step $x_{n+1} = x_n$. A large value for numIters must be chosen so that transient behavior which depends on the starting point is not displayed in the plot. Once the desired number of iterations has been done, another for-loop creates an array that stores the next numPoints number of iterations which are plotted. In this way, we see the stable equilibria, stable n-cycles, and chaotic behavior as such behaviors appear and disappear with increasing in r. It should be noted that the plots contained in this paper were made with IDL rather than the Java Applet.

We can see similar behavior in another similar function, $f(x) = rx(1-x^2)$ which satisfies the important conditions

1. $f(0) = f(1) = 0$
2. $f''(x) < 0$ for all $x \in [0,1]$
3. $f(x) \in [0,1]$ for all x.

2 Determining r_2

To formally define the custom dynamical system which will be used for most of this assignment let $f : [0,1] \to [0,1]$ by

$$f(x) = rx(1-x^2)$$

where $0 \leq r \leq \frac{1}{\sqrt{3}} \approx 2.59808$. The dynamical system is given by the rule $x_{n+1} = f(x_n)$ as before. In the bifurcation diagram, Figure 1, the behavior of the dynamical system changes with r. At some point, $r_2 \approx 2$, the stable equilibrium ceases to exist and bifurcates into a stable 2-cycle. The goal of this problem is to find r_2 precisely and determine the location of the fixed point just prior to the bifurcation.

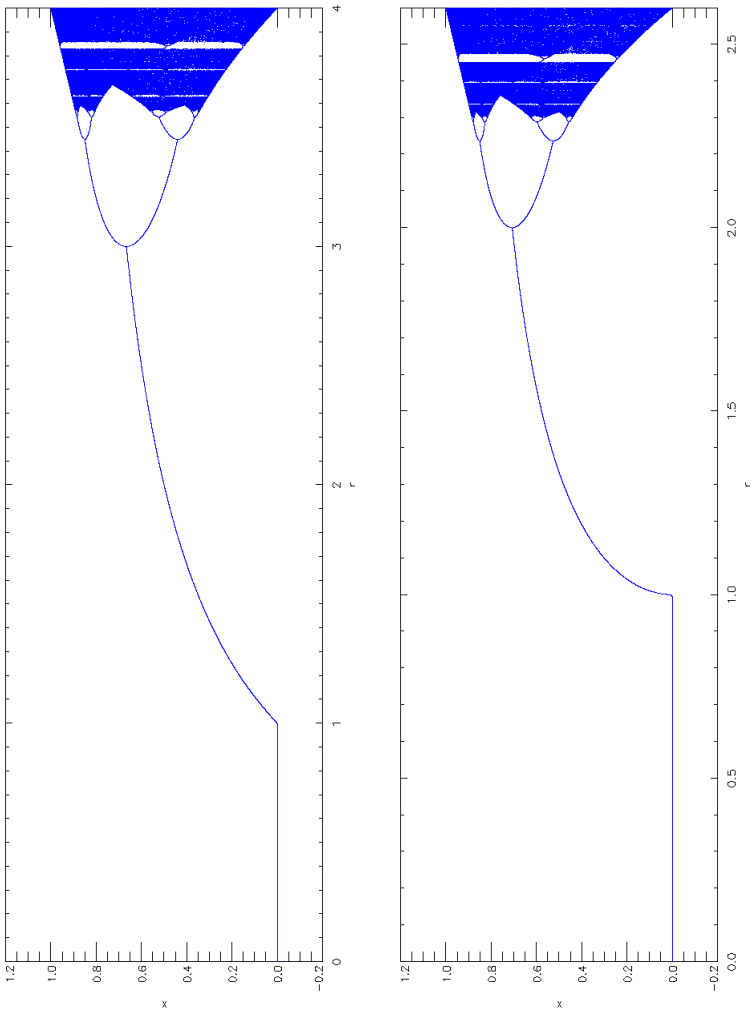
The stability of the single fixed point changes to unstable when the derivative at the fixed point has magnitude greater than one. So we are looking for the first value of r such that $|f'(x)| = 1$ at the fixed point x_r^*. The fixed point requirement is expressed by

$$\begin{aligned} x_{r_2}^* &= f(x_{r_2}^*) \\ &= r_2 x_{r_2}^* (1 - {x_{r_2}^*}^2) \end{aligned}$$

while the condition on the first derivative is

$$\begin{aligned} \pm 1 &= f'(x_{r_2}^*) \\ &= -2r_2 {x_{r_2}^*}^2 + r(1 - {x_{r_2}^*}^2). \end{aligned}$$

Figure 1: Bifurcation diagrams of the logistic map (left) and the custom function (right).



These can almost certainly be solved analytically, but it is simpler to make educated guesses based on numerics and then test them. From Figure 1, we can guess $r_2 = 2$. The Newton-Raphson method can then be applied to solve for the fixed point with this value for r. Again using the diagram, we can guess a starting value of $x = 0.7$ for the Newton-Raphson algorithm which yields $x^*_{r_2} \approx 0.7071$ which is very nearly $\sqrt{2}/2$. A quick analytic check confirms that the pair $x^*_{r_2} = \frac{\sqrt{2}}{2}$ are $r_2 = 2$ exactly satisfy the desired conditions so we're done! A Newton-Raphson based fixed-point finding applet was written as part of this assignment.

3 Finding a stable 2-cycle

A stable two-cycle may be found by identifying roots of the function $f^{(2)}(x) - x$ using the fixed point applet. As before, good guesses can be picked using the bifurcation diagram for say. The method works by calculating successive approximations of the root. It is generalized in the following way.

To calculate a fixed point of the function $f^{(n)}(x)$, first create an array of values, $\mathrm{xArray} = [x, f(x), f^{(2)}(x), ..., f^{(n-1)}(x)]$ and then calculate the derivative of f at each of these values, $f'(\mathrm{xArray})$. The product of all elements of the resulting array is $\frac{d}{dx} f^{(n)}(x)$ which can be used in the iterative step of a modified Newton-Raphson algorithm,

$$\begin{aligned} \triangle &:= -\frac{f^{(n)}(x) - x}{\frac{d}{dx} f^{(n)}(x) - 1} \\ x &:= x + \triangle. \end{aligned}$$

which calculates fixed points of $f^{(n)}(x)$ by finding roots of $f^{(n)}(x) - x$. This is applied to $f^{(2)}(x)$ with $r = 2.2$ in order to find the stable 2-cycle. The guesses chosen are 0.5 and 0.9 which give

Initial Guess, x	Estimated x^* of $f^{(2)}(x)$	Derivative at x^*	$f(x^*)$
0.6	0.54009359	-0.67992746	0.84160497
0.9	0.84160496	-0.67992746	0.54009363

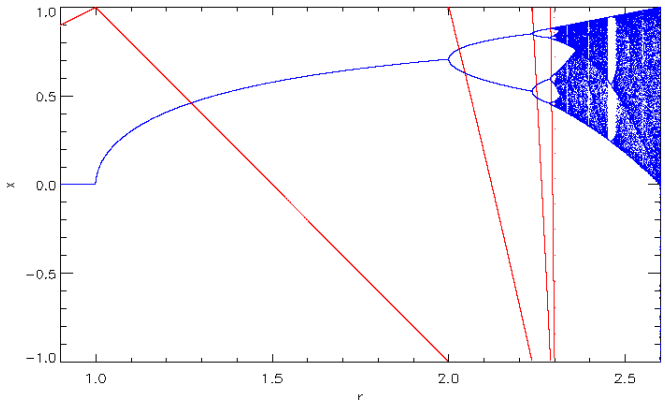
As can be seen, this is a stable 2-cycle since the derivatives have absolute value less than 1.

The same can be done in the 4-cycle arena at $r = 2.27$. With a guess of 0.4, we find an estimate for one member of a stable 2-cycle and can compute it more precisely by iterating f starting with the estimated member of the cycle, 0.46973689. After a few iterations, we find that the stable four cycle contains $0.4697368, 0.8310201, 0.5836675,$ and 0.8735659.

4 The Floquet Multiplier

In Chapter 3, we saw that the dynamical systems of the form being studied here experience a bifurcation when the Floquet multiplier becomes -1. This can be effectively

Figure 2: A bifurcation diagram along with the Floquet multipliers (in red). As expected, bifurcations occur when the multiplier reaches ± 1.



demonstrated by writing a program that plots the Floquet multiplier along with the bifurcation diagram as in Figure 2 which shows bifurcations when the multiplier approaches -1. The bifurcation diagram applet was modified to do just that.

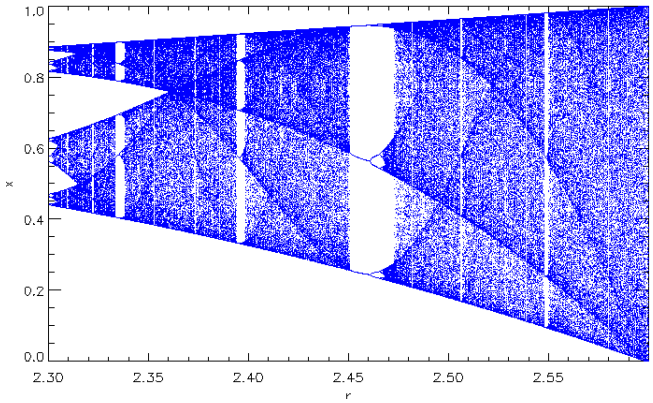
5 Changes of Behavior

If a the Bifurcation Diagram is blown up between $r = 2.3$ and $r = 2.6$, we can see some evident changes in behavior that were hinted at in the full diagrams. Taking a glance at Figure 3 we see that the chaos really goes into full mode when the cycles become infinite in length. This happens at $r_\infty \approx 2.30$. Not long after, the chaotic map seems to give way to n-cycle behavior once again. While this happens several times (I'm guessing an infinite number in actuality), the most clear is when a stable 3-cycle emerges at around $r = 2.45$. This well-behaved region ends at a new $r_\infty \approx 2.47$. There is another point of interest at around $r = 2.36$. Prior to this point, their is a clean separation between the upper and lower levels, but after it they appear to form a connected sub-space of $[0, 1]$!.

6 3-cycles in the Logistic Map

At this point, we return to the logistic map in order to investigate 3-cycles. Fortunately, these can be found using the fixed point finding applet mentioned previously.

Figure 3: Blown Up Bifurcation Diagram



3-cycles in the Logistic map, $r = 3.95$			
Initial Guess, x	Estimated x^* of $f^{(3)}(x)$	$f(x^*)$	Floquet Multiplier
0.0	0.0	0.0	61.629692
0.1	0.12396412	0.42895823	-6.1583283
0.2	0.18542999	0.59663056	6.7534044
0.4	0.42895823	0.96756463	-6.1583283
0.6	0.59663057	0.95061701	6.7534044
0.8	0.74683545	0.74683540	-7.4149206
0.9	0.95061702	0.18543001	6.7534044
1	0.96756462	0.12396406	-6.1583283

This indicates that we have two fixed points and two 3-cycles at $r = 3.95$ none of which are stable (since the multipliers all have absolute value greater than 1. The unstable fixed points are 0.0 and 0.74683545. The unstable cycles are

$$0.12396412 \to 0.42895823 \to 0.96756463 \to 0.12396406$$

and

$$0.18542999 \to 0.59663056 \to 0.95061701 \to 0.18543001.$$

Note that the three cycles can be grouped by multiplier value. If these values are perturbed a bit and the map iterated, it becomes clear that they are indeed unstable.

We can do the exact same thing for $r = 4$ and see how the 3-cycles change.

3-cycles in the Logistic map, $r = 4$			
Initial Guess, x	Estimated x^* of $f^{(3)}(x)$	$f(x^*)$	Floquet Multiplier
0.0	0.0	0.0	63.999808
0.1	0.11697778	0.41317592	−7.9999520
0.2	0.18825510	0.61126047	8.0000320
0.4	0.41317591	0.96984631	−7.9999520
0.6	0.61126047	0.95048443	8.0000320
0.8	0.75	0.75	−8.0000480
0.9	0.95048443	0.18825502	8.0000320
1	0.96984631	0.11697779	−7.9999520

As can be seen we still have two fixed points, 0.0 and 0.75 but in contrast to $r = 3.95$, the latter is a stable fixed point. We also have two unstable 3-cycles which can again be partitioned by Floquet multiplier.

$$0.11697778 \rightarrow 0.41317592 \rightarrow 0.96984631 \rightarrow 0.11697779$$

and

$$0.18825510 \rightarrow 0.61126047 \rightarrow 0.95048443 \rightarrow 0.18825502.$$

As r is decreased, all but two of the fixed points of $f^{(3)}(x)$ move toward one another and eventually annihilate. When this occurs, the 3-cycles no longer exist. This point can be found by trying to find the fixed point of $f^{(3)}(x)$ that lies at around 0.5 with the Newton-Raphson method. As r is decreased, this fixed point can no longer be found when r reaches 3.829. Indeed, none-of the three-cycles can be located indicating that they have ceased to exist.

7 Exact Solutions for N-Cycles at $R = 4$

It is possible to calculate exact values for N-Cycles at $R = 4$ using Equations 3.13-3.16. These can be modified slightly as follows. Used Equation 3.16 to find a member of an $N - 1$-cycle in its θ_i representation by setting

$$\begin{aligned} \theta_0 &= \theta_N - 2\pi \\ &= 2^N\theta_0 - 2\pi. \end{aligned}$$

From this relation, $\theta_0 = \frac{2\pi}{2^N-1}$. Then $\theta_1, \theta_2, ...\theta_{N-1}$ can be found from Equation 3.16. The standard x_i representation can then be found by modifying Equation 3.13,

$$x_i = \frac{1}{2}\left[1 - \cos\left(\theta_i\right)\right]$$

since the 2π is already included in the value of θ_0 here. We can then use this to locate say, a 4-cycle...

θ_i representation of a 4-cycle in the logistic map at $r = 34$.				
i	θ_i	x_i	$f(x_i)$	Floquet Multiplier
0	$2\pi/15$	0.04323	0.16544	16.00138
1	$4\pi/15$	0.16543	0.55225	16.00138
2	$8\pi/15$	0.55226	0.98908	16.00138
3	$16\pi/15$	0.98907	0.04324	16.00138

As can be seen from the Floquet multiplier, this is an unstable cycle.

The same can be done to search for a 5-cycle,

θ_i representation of a 5-cycle in the logistic map at $r = 34$.				
i	θ_i	x_i	$f(x_i)$	Floquet Multiplier
0	$2\pi/31$	0.01024	0.040541	32.00055
1	$4\pi/31$	0.04052	0.15551	32.00055
2	$8\pi/31$	0.15552	0.52533	32.00055
3	$16\pi/31$	0.52532	0.99743	32.00055
4	$32\pi/31$	0.99743	0.01025	32.00055

Once again, the cycle is unstable.

It should be noted that there are infinitely many cycles since N can be any natural number. Since the set $\left\{x_i = \frac{1}{2}\left[1 - \cos\left(\theta_i\right)\right] : \ \theta_i = {}^{2^{i+1}}/_{2^N-1}, \ i, N \in \mathbb{N}\right\}$ is dense in $[0, 1]$, the basin of attraction of each cycle is only a point because there is a point in another cycle located arbitrarily close. This demonstrates that for $r = 4$, all cycles are unstable! Admittedly, this is not a rigorous proof but rather gives the crucial point of the proof.

8 Sensitivity to Initial Conditions

A good notion of what the Chaos is like at various values of r can be attained by taking two initial values, x_0 and $x_0 + \epsilon$ where epsilon is small and iterating. The distance between their plots as a function of the number of iterations is then a good measure of how quickly differences in initial conditions lead to significant differences in the values taken on by the map. A program which does just this was written to create Figure 4. As can be seen, the distance initially increases with the number of iterations and then flits around the unit interval in a chaotic manner itself. Furthermore, decreasing ϵ increases the number of steps required to create significant distances. The practical implications for this in general chaotic systems is that predictability is quickly lost if one does not know the initial state with exact precision. Indeed, since computers carry out arithmetic with finite precision even perfect knowledge of initial conditions cannot save us.

Figure 4: A plot that shows the distance between two sequences, $f^{(n)}(x_0)$ and $f^{(n)}(x_0 + \epsilon)$ The upper plot shows the actual sequences while the lower shows the distance. The program was run to do this three different times. Both used $r = 4, x_0 = 0.4$, but ϵ was different for each. From top to bottom, we have $\epsilon = 0.001, \epsilon = 0.00001$, and $\epsilon = 0.0000001$. As can be seen, the number of steps taken to diverge significantly increases with decreasing ϵ and the distance initially follows a more or less exponential increase before becoming chaotic itself.

