Required Project III: Universality in Iterative Maps

In this project you will study in detail the properties of the sequence of period-doubling bifurcations of the maps such as the logistic map

$$x_{i+1} = f_r(x_i) = rx_i(1 - x_i). \tag{RP3.1}$$

In Required Project II you studied a few of these bifurcations by calculating the orbits of period 1, 2, and 4. Now you will examine the properties of the period-doubling sequence in detail. Chapter 4 of the class notes presents a strategy for computing orbits with very long periods. We expect you to use this strategy when you write the applets for this project probing the properties of these long-period orbits (unless, of course, you come up with a computational scheme that works even better!). The reason why it is interesting to study these long-period orbits is that they have universal scaling properties. First you will calculate the high-period orbits in the period-doubling sequence and find properties of these orbits that obey scaling laws. "Universality" means that these scaling laws are characterized by exponents that are exactly the same for a huge class of different mapping functions. You will explore this universality and understand why it arises by using the Feigenbaum equation. First you will demonstrate scaling empirically, by explicit computation of orbits in the period-doubling sequence.

Scaling in parameter space.

RP3.a. Write an applet that computes a sequence of r values, $\tilde{r}_0, \tilde{r}_1, ..., \tilde{r}_k$, for the superstable 2^k-cycles of the logistic map using Newton's method. You should be able to get as far as $k = 10$.

This sequence of r-values converges to an accumulation value, r_∞, at a geometric rate. That is,

$$\tilde{r}_k \simeq r_\infty - (\text{constant})\delta^{-k} \tag{RP3.2}$$

where δ is a number that we want to calculate. We can rewrite equation (RP3.2) as

$$\delta \simeq \frac{\tilde{r}_{k-1} - \tilde{r}_{k-2}}{\tilde{r}_k - \tilde{r}_{k-1}} \tag{RP3.3}$$

(Please verify this by substituting from Equation (RP3.2).)

Since you have already computed the $\tilde{r}_k$'s, it is a simple matter to plug them into equation (RP3.3) to get an estimate of δ. Please have your applet do so.

Scaling in state space.

In Equation (RP3.1) you calculated how the parameter value r changes as one goes from a 2^k to a 2^{k+1} superstable cycle—it gets scaled by a factor δ. This is a scaling in "parameter space," since r is a parameter of the map. Now you will characterize the scaling in x (which is called "state space," since x describes the state of the system). Again consider the superstable 2^k cycles. Every superstable cycle must contain $x = 1/2$, the point where the function reaches its maximum value (and thus its derivative is zero). Now we ask, for a cycle of length 2^k that starts at $x = 1/2$, what is $f_{\tilde{r}_k}^{(2^{k-1})}(x = \frac{1}{2})$, the value of x exactly halfway through the cycle?

RP3.b. *optional.* We claim that $f_{\tilde{r}_k}^{(2^{k-1})}(x = \frac{1}{2})$ is closer to 1/2 than any other point in the orbit. Please write an applet that checks this claim numerically for a few values of k.

RP3.c. Have your applet compute the $y_k = x_{2^{k-1}} = f_{\tilde{r}_k}^{(2^{k-1})}(x = \frac{1}{2})$ for $k = 2$ through 10. You should find that the y_k's also converge geometrically with k:

$$\frac{1}{2} - y_k \simeq (-\alpha)^{-k}, \tag{RP3.4}$$

where

$$-\alpha \simeq \frac{y_{k-1} - y_{k-2}}{y_k - y_{k-1}}. \tag{RP3.5}$$

Please find α.

Notice that $-\alpha$ is negative. This means that these points are falling on alternate sides of $x = 1/2$. You may confirm this behavior by looking at the bifurcation diagram.

RP3.d. *optional.* Write an applet that generates the series x_i for the logistic map with $r = r_\infty$, starting with $x_0 = x_m(= \frac{1}{2})$ (it should be easy to modify your applet from Part c to do this). Let $\ell = 2^k \ell_0$ (for example, 3, 6, 12, 24, ... or 5, 10, 20, 40...). Demonstrate that the x's near the maximum $x = \frac{1}{2}$ appear to obey the equation

$$-\alpha \left(x_{2\ell} - \frac{1}{2} \right) = x_\ell - \frac{1}{2}, \tag{RP3.6}$$

with the same value of α found above.

Universality.

Now we ask you to look at the question of universality empirically by computing the exponents for the period-doubling sequences of other functions. Consider the following functions:

$$F_r = rx \sin(\pi x) \quad \text{for} \quad 0 < x < 1; \quad 0 < r < 1.7 \tag{RP3.7}$$

$$G_r = rx^2 \sqrt{1 - x} \quad \text{for} \quad 0 < x < 1 \tag{RP3.8}$$

Both these functions have single humps*; (RP3.7) has its maximum at x near 0.65, while (RP3.8) has its maximum at $x = 4/5$

RP3.e. Choose one of the functions above, (RP3.7) or (RP3.8), and, using modified versions of the programs you used for the logistic map: (a) Draw the bifurcation diagram. (b) Compute α and δ. Note that the scaling no longer occurs near $x = 0.5$ but at the x where F_r or G_r are maximal.

* Both of these functions have quadratic maxima. You can verify this for yourself by taking two derivatives of each function, and plugging in the x value at the maximum (*i.e.,* the top of the hump).

The renormalization group.

The class notes argue that an arbitrary function with a period-doubling sequence and a maximum at x_m converges after many iterations to a function $g(z)$ obeying the Feigenbaum Equation:

$$g(\beta z) = \beta g(g(z)) \quad \text{or} \quad g(z) = -\alpha g(g(-z/\alpha)), \tag{RP3.9}$$

where $z \propto (x - x_m)$.

RP3.f. In the class notes the exponent α characterizing the period-doubling route to chaos is calculated for functions which have a quadratic maximum (the generic case, which includes the logistic map). Here you will calculate the exponents that characterize the period-doubling route to chaos for a function with a particular type of cusp, specifically

$$f(x) = r\left((\frac{1}{2})^{3/2} - |x - \frac{1}{2}|^{3/2}\right).$$

This function also exhibits period doubling. Thus the reasoning of Chapter 5 leading to the Feigenbaum equation (RP3.9) still holds.
(a) Estimate the values of α and δ for this map using the method you used for the logistic map in (RP3.1) and (RP3.3) (*i.e.,* , by calculating the sequence of superstable 2^k cycles, and finding δ from the $\tilde{r}_k$ values and α from distance that the (2^{k-1} th iterate starting from 0 is from $z = 0$).
(b) For functions with a 3/2-power cusp, we can expand $g(z)$ near z=0 as:

$$g(z) = A - B|z|^{3/2} + Cz^2 + ...,$$

and solve the Feigenbaum equation approximately by ignoring terms of order higher than $|z|^{3/2}$. Find an approximate value for α using this method. (Hint: When you equate the coefficients of the powers of z, you will find an algebraic equation that α must satisfy. You may solve this equation numerically using your Newton-Raphson routine, or a package such as Maple, Matlab, or Mathematica.) How does it compare with the result you obtained by direct computation of the orbits? How would you improve the accuracy of your approximate solution to the Feigenbaum equation?