Physics 316, Problem Set 1, due Tuesday October 4 in class

1. Schwarz's inequality: A vector space V has a norm, with the properties listed in the Glossary (see `materials` in web site). Show that its natural scalar product $(\mathbf{x},\mathbf{y})$ satisfies $(\mathbf{x},\mathbf{y})^2 \le (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})$. This is known as the Schwartz inequality.

Solution:

$$(\mathbf{x},\mathbf{y}) = \frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right)$$

We observe that

$$(\mathbf{x},\mathbf{x}) = \|\mathbf{x}\|^2$$

for any $\mathbf{x}$. According to the triangle inequality,

$$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$$

$$\|\mathbf{x}+\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\|$$

Expressing $\|\mathbf{x}+\mathbf{y}\|$ using the definition of $(\mathbf{x},\mathbf{y})$,

$$2(\mathbf{x},\mathbf{y}) + \|\mathbf{x}\|^2 + \|\mathbf{x}\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\|$$

$$2(\mathbf{x},\mathbf{y}) \le 2\|\mathbf{x}\|\,\|\mathbf{y}\|$$

dividing by 2 and squaring...

$$(\mathbf{x},\mathbf{y})^2 \le (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})$$

QED

2. Affine transformations and linear transformations: A certain affine space is also a vector space, and a line in this vector space is given by $\mathbf{y}(\eta) \equiv \mathbf{b} + \eta\mathbf{x}$.

a) Find the analogous expression for a line passing through the origin ($\mathbf{0}$) of the vector space that is parallel to the given line .

b) A transformation $f : \mathbf{x} \in \mathbb{R}^2 \to \mathbf{y} \in \mathbb{R}^2$ in a two-dimensional space $\mathbb{R}^2$ is defined by

$$(y_1, y_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (x_1, x_2) + (g_1, g_2),$$

where $a...d$, g_1 and g_2 are constants. Show that the image of the generic line defined in a) is also a line.

Solution:

a) Let's call the desired line passing through the origin $\mathbf{z}(\eta)$. If $\mathbf{z} = \mathbf{b}' + \eta\mathbf{x}'$ includes the origin, then $\mathbf{b}'$ must be $\mathbf{0}$. To make $\mathbf{z}$ be parallel to $\mathbf{y}$ we can take $\mathbf{x}' = \mathbf{x}$: there is no value of η, η' such that $\mathbf{y}(\eta) = \mathbf{z}(\eta')$ thus the two lines don't intersect. The same is true if $\mathbf{x}'$ is a scalar multiple of $\mathbf{x}$.

b) Since $\mathbf{x}(\eta)$ is a line, it can be expressed $\mathbf{x}(\eta) = \mathbf{b} + \eta\mathbf{u}$. Using the component language $\mathbf{x} = (x_1, x_2)$, this means $(x_1, x_2)(\eta) = (b_1 + \eta u_1, b_2 + \eta u_2)$. We need to show that $(y_1, y_2)(\eta)$ can be expressed in the form $\mathbf{B} + \eta\mathbf{U}$, or in component language, $(y_1, y_2) = (B_1 + \eta U_1, B_2 + \eta U_2)$. On the other hand the formula says

$$(y_1, y_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (b_1 + \eta u_1, b_2 + \eta u_2) + (g_1, g_2)$$

or

$$(y_1, y_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (b_1, b_2) + \eta \begin{bmatrix} a & b \\ c & d \end{bmatrix} (u_1, u_2) + (g_1, g_2)$$

We see that $\mathbf{y}$ is the sum of a fixed vector $\mathbf{B}$ and a second vector $\mathbf{U}$ times η, where

$$\mathbf{B} = (B_1, B_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (b_1, b_2) + (g_1, g_2)$$

and

$$\mathbf{U} = (U_1, U_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (u_1, u_2)$$

3. Three points in a plane: Arnold, p. 10, third problem

Solution:

We denote the particle orbits by $\mathbf{a}(t), \mathbf{b}(t)$, and $\mathbf{c}(t)$. We may define a basis for displacements in this plane by $\mathbf{b}(0) - \mathbf{a}(0)$ and $\mathbf{c}(0) - \mathbf{a}(0)$. We are supposed to show that the orbits lie in this plane. To show that $\mathbf{a}(t)$ lies in it, we have to show that $\mathbf{a}(t) = \mathbf{a}(0) + (\text{multiple of})(\mathbf{b}(0) - \mathbf{a}(0)) + (\text{multiple of})(\mathbf{c}(0) - \mathbf{a}(0))$. We do this by showing that the accelerations $\ddot{\mathbf{a}}, \ddot{\mathbf{b}}$, and $\ddot{\mathbf{c}}$ lie in the plane. We first consider $\ddot{a}$ at $t = 0$. This acceleration must be a deterministic function of positions and velocities:

$$\ddot{\mathbf{a}}(0) = F_a(\mathbf{a}(0), \mathbf{b}(0), \mathbf{c}(0), \dot{\mathbf{a}}(0), \dot{\mathbf{b}}(0), \dot{\mathbf{c}}(0))$$

The three time derivitives are zero, by hypothesis. We recall that for any Galilean transformation $\mathbf{g}$, the function F_a must satisfy

$$\mathbf{g}F_a(\mathbf{a}(0), \mathbf{b}(0), \mathbf{c}(0), \dot{\mathbf{a}}(0), \dot{\mathbf{b}}(0), \dot{\mathbf{c}}(0)) = F_a(\mathbf{g}\mathbf{a}(0), \mathbf{g}\mathbf{b}(0), \mathbf{g}\mathbf{c}(0), \mathbf{g}\dot{\mathbf{a}}(0), \mathbf{g}\dot{\mathbf{b}}(0), \mathbf{g}\dot{\mathbf{c}}(0))$$

i.e., if we transform the co-ordinates and velocities, the acceleration must undergo the same transformation. We use this freedom to simplify the problem. We choose $\mathbf{g}$ to be a translation that shifts $\mathbf{a}$ to the origin. This has no affect on velocities or accelerations:

$$\ddot{\mathbf{a}}(0) = F_a(0, \mathbf{b}(0) - \mathbf{a}(0), \mathbf{c}(0) - \mathbf{a}(0), 0, 0, 0)$$

We now take $\mathbf{g}$ to be an orthogonal transformation that leaves the every vector in the $(\mathbf{b} - \mathbf{a})$ — $(\mathbf{c} - \mathbf{a})$ plane invariant. In particular $(\mathbf{g}\mathbf{b} - \mathbf{g}\mathbf{a}) = (\mathbf{b} - \mathbf{a})$, and $(\mathbf{g}\mathbf{c} - \mathbf{g}\mathbf{a}) = (\mathbf{c} - \mathbf{a})$. This transformation is a reflection in the plane. Using this transformation

$$\mathbf{g}\ddot{\mathbf{a}}(0) = F_a(0, \mathbf{b}(0) - \mathbf{a}(0), \mathbf{c}(0) - \mathbf{a}(0), 0, 0, 0) = \ddot{\mathbf{a}}(0)$$

This transformation cannot change the acceleration. But the only vectors that are unchanged in this translation are precisely the ones in the $(\mathbf{b} - \mathbf{a})$ — $(\mathbf{c} - \mathbf{a})$ plane. Thus $\ddot{\mathbf{a}}(0)$ must lie in this plane. The same argument shows that $\ddot{\mathbf{b}}(0)$ and $\ddot{\mathbf{c}}(0)$ must lie in this same plane.

Since the initial accelerations are in the plane, the initial velocities must be proportional to t and must lie in the same plane. We now consider the acceleration $\ddot{\mathbf{a}}(t)$ for a time when $\dot{\mathbf{a}}(t)$, $\dot{\mathbf{b}}(t)$, and $\dot{\mathbf{c}}(t)$ lie in the plane. Our reflection transformation evidently leaves $\dot{\mathbf{b}}(t)$, and $\dot{\mathbf{c}}(t)$ invariant, just as it did $(\mathbf{b} - \mathbf{a})$ and $(\mathbf{c} - \mathbf{a})$. Thus again $\ddot{\mathbf{a}}$ cannot change under the transformation, and so $\ddot{\mathbf{a}}(t)$ must be in the plane. The same is true of $\ddot{\mathbf{b}}(t)$ and $\ddot{\mathbf{c}}(t)$, as before. We conclude that the velocities must remain in the plane since they never acquire an acceleration that takes them out of it. Finally, since the velocities remain in the plane, the displacements $\mathbf{a}(t) - \mathbf{a}(0) = \int_0^t \dot{\mathbf{a}}(t')dt'$ must also lie in the plane. This is what we were supposed to show.

4. Two points in a plane: Arnold, p. 10, fourth problem.

Solution: We let the two orbits be $\mathbf{a}(t)$ and $\mathbf{b}(t)$, as above. We may simplify the math by translating $\mathbf{a}(0)$ to the origin as in the previous problem. We may also move to a frame at rest with the initial a particle. Then

$$\ddot{\mathbf{a}}(t) = F_a(0, (\mathbf{b}(t) - \mathbf{a}(0)), \dot{\mathbf{a}}(t) - \dot{\mathbf{a}}(0), \dot{\mathbf{b}}(t) - \dot{\mathbf{a}}(0))$$

We first show that $\ddot{\mathbf{a}}(0)$ must lie in the plane defined by $(\mathbf{b}(0) - \mathbf{a}(0))$ and $\dot{\mathbf{b}}(0)$. We do this by applying the reflection transformation $\mathbf{g}$ as in the previous problem. The same argument shows that $\ddot{\mathbf{b}}(0)$ must lie in this same plane. Thus subsequent initial velocities of both a and b lie in this plane. Whenever the velocities and displacements lie in this plane accelerations must also lie in it, using the same reflection transformation. But the velocities and displacements *do* stay in this plane, since the accelerations do.

5. Phase flow in one dimension: When a particle moves in one dimension x in a potential $U(x)$, the resulting motion over a very short time interval is specified by Newton's equation, provided x and $\dot{x}$ are

given. Often the resulting motion can be determined over an arbitrarily long time. The resulting motion in phase space is called the *phase flow*. But for some potentials the motion can't be determined to arbitrarily long times: there is no phase flow. If $U(x)$ has the form $(-\text{const } x^n)$, there are some values of the positive integer n for which the motion doesn't define a phase flow. What are these values and what prevents the system from having a phase flow? (*cf.* Arnold, p. 21 first problem) .

Solution: We attempt to predict $x(t)$ for all time and see when we get into trouble.

$$\frac{1}{2}\dot{x}^2 - Ax^n = E$$

$$\dot{x} = 2^{1/2}(E + Ax^n)^{1/2}$$

$$\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{dx'}{2^{1/2}(E - A(x')^n)^{1/2}}$$

Let $u = Ax^n$; then $x = A^{-1/n}u^{1/n}$; $dx = A^{-1/n}(1/n)u^{1/n-1}\ du$. Then

$$(t - t_0) = \frac{A^{-1/n}}{2^{1/2}n}\int_{u_0}^{u} du' \frac{(u')^{1/n-1}}{(E + u')^{1/2}}$$

In order to have a phase flow, the equation must be satisfied for some u as $t \to \infty$. The integral on the right must thus diverge. This is only possible if the upper limit goes to infinity. However, for large u' (*i.e.,* large x) the integrand reduces to $\frac{(u')^{1/n-1}}{(u')^{1/2}} = (u')^{1/n-3/2}$. This power is less than -1 whenever $n > 2$; then the integral cannot become infinite. Then there is a finite value of t for which $u \to \infty$, and motion is not defined beyond this time. Thus for any integer n greater than 2, there is no phase flow.