Physics 316, Problem Set 2, due Tuesday October 11 in class

1. Phase space area: For one dimensional periodic motion in an external potential, with energy E, let $S(E)$ be the area enclosed by the phase curve during one period.

a)Show that the period T (the time required for the motion to repeat) is given by dS/dE. (*cf.* Problem p. 20)

Suppose T varies smoothly with E. If the initial position x_0 is displaced infinitessimally, the displaced orbit $x_d(t)$ is altered from the unperturbed orbit $x(t)$ Moreover, $(x_d(t) - x(t)) \equiv \Delta(t)$ grows with time. If $\Delta(0)$ is very small, it must remain much smaller than the original orbit for many orbital periods even though it may grow by a large factor. The problem concerns how Δ grows during this range of time.

b) Express the shift in energy $E_d - E$ as a function of $\Delta(0)$ and the potential energy $U(x)$.

c) Determine how Δ grows by considering times t that are large integer multiples of the original period T as a function of dT/dE. Consider times such that Δ remains small compared to the original orbit.

Solution: a) The area enclosed by the phase curve during one period can be expressed as:

$$S(E) = 2\int_{x_{min}}^{x_{max}} \dot{x}dx = 2\int_{x_{min}}^{x_{max}} \sqrt{2(E-U)}dx.$$

And the period can be expressed as:

$$T = 2\int_{x_{min}}^{x_{max}} \frac{dx}{\dot{x}} = 2\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{2(E-U)}}.$$

So

$$\frac{dS}{dE} = 2\int_{x_{min}}^{x_{max}} \frac{1}{\sqrt{2(E-U)}}dx = T.$$

b) Assume the initial position is infinitesimally displaced to $y_0 = x_0 + \Delta_0$ with $\Delta_0 << x_0$ and the period is correspondingly shifted to $T_y = T + \frac{dT}{dE}dE = T + \frac{dT}{dE}\frac{dU}{dx}|_{x_0}\Delta_0$. Since both are periodic motions, we have for generic point x and its corresponding y that $x(nT) = x(0)$ and $y(nT_y) = x(0) + \Delta$. Notice that since it's a infinitesimal change, we have $\Delta_0 = \Delta + O(\Delta^2)$.

Using Taylor expansion for $y(nT)$, we get:

$$y(nT) = y(nT_y) + \dot{y}(nT_y)\cdot n(T - T_y) + \ldots \approx y(nT_y) + \dot{y}(nT_y)[n(T - T_y)]$$

i.e.

$$y(nT) = (x(0) + \Delta) + \dot{y}(nT_y)\left[n\frac{dT}{dE}\left.\frac{-dU}{dx}\right|_{x_0}\Delta_0\right] = (x(nT) + \Delta) + \dot{y}(0)\left[n\frac{dT}{dE}\left.\frac{-dU}{dx}\right|_{x_0}\Delta_0\right].$$

Noting that $\dot{y}(0) = \dot{x}(0) + O(\Delta)$ and $\Delta_0 = \Delta + O(\Delta^2)$,

$$y(nT) - x(nT) = \Delta + \dot{x}(0)\left[n\frac{dT}{dE}\left.\frac{-dU}{dx}\right|_{x_0}\Delta\right]$$

In other words which can be simplified to

$$\frac{d\Delta}{dt} = \frac{1}{T}\frac{d\Delta}{dn} = \frac{1}{T}\left[\dot{x}(0)\frac{dT}{dE}\left.\frac{-dU}{dx}\right|_{x_0}\right]\Delta$$

The distance between x and y thus grows exponentially in time at a rate controlled by dT/dE.

2. Central motion review problem: A particle of unit mass moves in an attractive central potential $U(r) = -k\exp(-r/r_0)$. Its orbit is nearly circular with radius R.

a) Find the angle Φ between points of maximum and minimum r for arbitarary orbit radius R using methods of Chapter 2. Explain the behavior for large R.

b) Find values of R such that the orbit closes after n revolutions.

Solution:

a) The angle Φ between r_{max} and r_{min} for an orbit close to the circle of radius R is (cf. Problem 2 on P 37):

$$\Phi = \pi\sqrt{U'(r=R)/(3U'(r=R) + RU''(r=R))}$$

where $V(r) = U(r) + \frac{M^2}{2r^2}$ is the effective potential.

For this problem, we have $U(r) = -k\exp(-r/r_0)$; $U' = -U/r_0$; $U'' = U/r_0^2$. Thus we get $\Phi = \frac{\pi}{\sqrt{3-R/r_0}}$. If $R > 3r_0$, there is no solution for Φ. Here the second derivitive of the effective potential $V(r)$ is negative near R and there are no small oscillations around R.

b) The condition for a closed orbit after n revolutions is $2n\Phi = 2m\pi$, i.e. $R = r_0(3 - (n/m)^2)$.

3. Euler Lagrange equation for fancier functionals Given a function $\mathcal{L}$ of three variables, we may define a functional $F[x(t)]$ for functions $x(t)$ on the unit interval $0 \le t \le 1$ by:

$$F[x] \equiv \int_0^1 dt\ \mathcal{L}(x(t), \dot{x}(t), \ddot{x}(t)).$$

Find an expression for the linear variation δF when $x(t)$ is perturbed by a small test function $h(t)$. From this, find the differential equation analogous to the Euler Lagrange equation for functions $x(t)$ that have $\delta F = 0$ for arbitrary test functions $h(t)$.

Solution: Using the variational method, we have:

$$\delta F = \delta[\int_0^l L(x, \dot{x}, \ddot{x})dt] = \int_0^l dt[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial \ddot{x}}\delta\ddot{x}].$$

Integrating by parts, we get:

$$\delta F = \int_0^l dt\delta x\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right] + \left[\delta x\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)\right]_0^l + \left[\delta\dot{x}\frac{\partial L}{\partial \ddot{x}}\right]_0^l.$$

Since the integral must vanish for all test functions $\delta x(t)$, the coefficient of δx in the integrand must vanish. Thus

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0$$

This is the analog of the Euler Lagrange equation for this system.

4. Bending rod

A flexible, massless rod of length L is mounted vertically at one end as shown. From the other end is suspended a weight W, causing the rod to bend. The rod's shape may be specified by giving the angle $\theta(s)$ at distance s from the base. The bending energy K of the rod is proportional to the square of the curvature:

$$K = \frac{1}{2}B\int_0^L ds\left(\frac{d\theta}{ds}\right)^2,$$

where the constant B is the bending stiffness. The rod adopts the shape that minimizes the total energy .

a) Find a differential equation satisfied by $\theta(s)$.

b) Find $d\theta/ds|_{s=L}$.

c) Find a function of θ and $d\theta/ds$ that is constant for all s—*i.e.,* a "constant of the motion."

d) Determine the value of this constant if the weighted end is horizontal.

e) Find the length L which makes the weighted end horizontal for the given weight W.

Solution:

a) The energy $\mathcal{E}$ of the system is extremal. We may express the energy in terms of $\theta(s)$. To express the energy due to the weight, we note that the height h of the tip can be written $h = \int ds\cos\theta(s)$. Then the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \int_0^L ds\left[B\dot\theta^2 + W\cos\theta(s)\right]$$

where I defined $\dot\theta \equiv d\theta/ds$. The integrand is a functional of θ and $\dot\theta$; it has the form of a Lagrangian. Accordingly, we denote it by $\mathcal{L}$. The extremal $\theta(s)$ satisfies the Euler-Lagrange equation

$$\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

For our case this amounts to

$$B\ddot\theta + W\sin\theta = 0$$

The rod is formally equivalent to a physical pendulum.

b) $\dot\theta = 0$ because otherwise the energy is not stationary with respect to variations $\dot h(s = L)$.

c) Since this is a Lagrangian mechanics problem, we look for constants of the motion as found in such problems. Since there is no explicit dependence on the "time" variable s, the Hamiltonian $E \equiv \mathcal{L} - \dot\theta\partial\mathcal{L}/\partial\dot\theta$ must be conserved. (E has to be just the total energy of the analogous physical pendulum.) This $\mathcal{L}$ has the usual form of a kinetic energy quadratic in $\dot\theta$ plus a potential energy independent of $\dot\theta$, *viz.* $W\cos\theta$. Then $E = \mathcal{L} - 2W\cos\theta = \frac{1}{2}B\dot\theta^2 - W\cos\theta$. The conventional interpretation of this condition is that torques on the segment of the rod from 0 to s must balance.

d) At the weighted end, we found that $\dot\theta = 0$. We are told that the rod is horizontal, so that $\theta = \pi/2$. Applying the above formula for E, evidently $E = 0$.

e) Since this is a one-dimensional mechanics problem, the trajectory $\theta(s)$ can be reduced to an integral, specifically, since $\frac{1}{2}B\dot\theta^2 - W\cos\theta = E$, we have

$$\dot\theta = \sqrt{(E + 2W\cos\theta)/B}$$

so that

$$L = \int ds = \int (ds/d\theta)d\theta = \int_0^{\pi/2}\frac{d\theta}{\sqrt{(E + 2W\cos\theta)/B}}$$

where we used the given fact that $\theta = \pi/2$ at the free end. Thus, using $E = 0$

$$L\sqrt{\frac{W}{B}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}} = \frac{\sqrt{\pi}\,\gamma(\frac{1}{4})}{2\,\gamma(\frac{3}{4})}$$