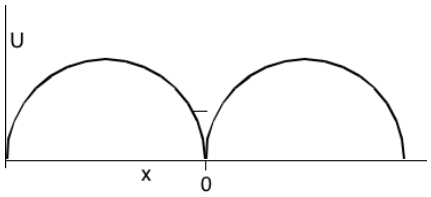
Physics 316, Problem Set 3, due Tuesday October 18 in class

1. method of similarity: (*cf.* Chapter 2 Section 11)



The text demonstrates a way of inferring dynamical behavior by analyzing the scaling properties of the equation without solving it. This problem is meant to show these procedures. Here we wish to find the limiting properties of a system's motion as the energy of a particle goes to zero. The potential energy function $U(x)$ of a point particle has the form of two semicircles side by side: $U(x) = U_0\sqrt{1-(1-|x|)^2}$. A particle with energy $E \ll U_0$ is trapped in the region near $x = 0$. It is obliged to oscillate with some period τ. The object is to find how the period τ varies as the energy $E \to 0$ by the method of similarity. That is, define scaled variables $\tilde{E} \equiv \lambda E$, $\tilde{t} \equiv \mu t$ and $\tilde{x} \equiv \eta x$. Then determine how μ and η must vary with λ in order to leave the equation of motion invariant.

a) How do the maximum displacement in x and the temporal period τ vary with E as $E \to 0$?

Solution: In the small-x region where the particle is trapped the equation for conservation of energy may be approximated

$$\frac{1}{2}m\frac{d^2x}{dt^2} + U_0\sqrt{2|x|}(1-\frac{1}{4}|x|) = E$$

As in the text we define ˜ variables to represent the rescaled quantities which are to remain finite as the energy E goes to zero. $\tilde{E} \equiv \lambda E$; $\tilde{t} \equiv \mu(\lambda)t$; $\tilde{x} \equiv \eta(\lambda)x$. In terms of these variables

$$\frac{1}{2}m\left(\frac{d\tilde{x}\eta^{-1}}{d\tilde{t}\mu^{-1}}\right)^2 + U_0\sqrt{2|\tilde{x}|\eta^{-1}}(1-\frac{1}{4}|\tilde{x}\eta^{-1}|) = \tilde{E}\lambda^{-1}$$

combining the scale factors,

$$\frac{1}{2}m\frac{\lambda\mu^2}{\eta^2}\left(\frac{d\tilde{x}}{d\tilde{t}}\right)^2 + U_0\lambda\eta^{-1/2}\sqrt{2|\tilde{x}|} - \sqrt{2}\frac{U_0}{4}\lambda\eta^{-3/2}|\tilde{x}|^{3/2} = \tilde{E}.$$

To make the second term be finite, we need $\eta \sim \lambda^2$. Then to make the first term finite, we need $\lambda\mu^2/\eta^2 = \lambda\mu^2/\lambda^4 = \lambda^{-3}\mu^2$ finite. Thus $\mu \sim \lambda^{3/2} \sim \eta^{3/4}$. In this situation, the last term starting with $\sqrt{2}$ goes to zero, so the equation is consistent to neglect it. If we had tried to make the last term stay finite, by setting $\eta \sim \lambda^{2/3}$, then the second term would have been infinite, and the equation would not be tractible.

Denote the maximum x displacement as x_{max}. Now, $x_{max} = \tilde{x}_{max}\eta^{-1} \sim \tilde{x}_{max}\lambda^{-2} \sim \tilde{x}_{max}(\tilde{E}/E)^{-2}$. By construction, the ˜ variables stay finite; thus $x_{max} \sim E^2$ as $E \to 0$. By parallel reasoning, the period τ must scale along with other times, so that $\tilde{\tau} = \tau\mu$ remains finite. Since $\mu \sim \lambda^{3/2}$, we find $\tau \sim E^{3/2}$.

2. Legendre transform of simple functions

a) Find the Legendre transform of $f(x) \equiv \frac{1}{m}x^m$, $f(x) \equiv e^{\lambda x}$, and $f(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$. For the last case, not all x's and p's are allowed.

b) Graph $f(x)$ and $g(p)$ for the last two examples from part a). Does g(p) behave as expected at the ends of its range?

Solution: SKETCH of SOLUTION power: $g(p) = \frac{m-1}{m}p^{m/(m-1)}$ exponential: $g(p) = -p\log p$ except for coefficients. minimal p is p=0. There $x \to -\infty$, so dg/dp must go to $-\infty$. The largest p is $+\infty$, and there dg/dp goes to $+\infty$, as it should, but very slowly. binomial: to find $x(p)$ must solve a quadratic equation. $g(p)$ is a nasty function, but it should have a quadatic minimum at $p = 0$ and behave as $p^{3/2}$ for large p.

Since $d^2f/dx^2 > 0$, we have $x > -1/2$

$p = df/dx = x + x^2$, and $x = (\sqrt{(1+4p}-1)/2$, with $p > -1/4$
And

$$g(p) = px - f(x) = xp - 1/2x^2 - 1/3x^3$$

(Using $x + x^2 = p$)

$$= px - 1/2x^2 + 1/3x^2 - 1/3(x^3 + x^2)$$

$$= px - 1/6x^2 - 1/3xp$$

$$= 2/3xp - 1/6x^2$$

$$= 2/3xp - 1/6(p - x)$$

$$= -1/6p + x(2/3p + 1/6)$$

$$= -1/6p + (2/3p + 1/6)(sqrt(1 + 4p) - 1)/2)$$

$$= (1 + p)/2 + 1/12(1 + 4p)^3/2p > -1/4$$

3. Legendre Transform properties

a) If $f(x)$ has Legendre transform $g(p)$, what is the Legendre transform of $f(x) + bx$?
b) If $f_0(x)$ has Legendre transform $g_0(p)$, what is the Legendre transform of $f(x) = \lambda f_0(x) + c$? λ and c are constants.
c) If $f_0(x)$ has Legendre transform $g_0(p)$, what is the Legendre transform of $f(x) = f_0(\lambda x + c)$?

SKETCH of SOLUTION All these use the definition $g(p) = (px - f(x))_{x\text{such that}_{p=f'(x)}}$, expressing f in terms of f_0, together with product rule, substitutions.

a) $\partial f/\partial c_x = \partial g/\partial c_p$
b) $g(p) = g_0(p + b)$
c) $g(p) = \lambda g_0(p/\lambda) - c$
d) $g(p) = g_0(p/\lambda)$.

4. When is Hamiltonian = energy In a strange world without Galilean invariance, Newton's laws have a different form. In this world a certain one-particle system has a Lagrangian $\mathcal{L}$ with the form $\mathcal{L} = \frac{1}{4}m\dot{x}^4 - U(x)$, where x is the position co-ordinate.

a) Find the generalized momentum p for this system
b) Find the Hamiltonian.
c) Is the "energy" $\frac{1}{4}m\dot{x}^4 + U(x)$ conserved? Is the Hamiltonian conserved?

Solution: $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}^3$ $\mathcal{H} = \frac{3}{4}m^{-1/3}p^{4/3} + U(x)$ $\mathcal{H}$ is conserved, since it does not depend explicitly on time. $E = \frac{1}{4}m\dot{x}^4 + U(x)$ is not conserved.