Physics 316, Problem Set 5, due Tuesday November 1 in class

1. Noether's theorem for a snakey rope A flexible rope of fixed length L has a mass-per-unit-length ρ_l, and negligible bending energy. It is free to move in space with no gravity. It is charged along its length with uniform linear density σ so that it repels itelf. Its configuration in space is labeled by the function $\mathbf{r}(u)$, where u is the position along the rope.

a) What is the Lagrangian of this system?

b) Now suppose that the rope has the form of a uniform loop of an arbitrary shape. Use Noether's theorem to find a functional of $r(u)$ and $\dot{r}(u)$ that is a constant of the motion.

Solution: a)The Lagrangian is (in SI units):

$$L = \int_0^L \frac{1}{2}\rho_l[\dot{\vec{r}}(u)]^2 du + \frac{1}{2}\int du_1 \int du_2 \; \sigma^2 (4\pi\epsilon_0)^{-1} \left|\mathbf{r}(u_1) - \mathbf{r}(u_2)\right|^{-1}.$$

b) We can easily get the conservation of energy, angular momentum, etc. But we are most interested in the conserved quantity that can't be easily guessed but can be obtained by Noether's theorem.

Since the rope forms a loop, the Lagrangian is unchanged by the mapping: $h^s(\vec{r}(u)) = \vec{r}(u+s)$. By Noether's theorem, we have a corresponding conserved quantity. But we have to be careful here because the form of Noether's theorem in the textbook is for finite number of local coordinates q and what we have now is an infinite number of local coordinates $\vec{r}(u)$ on the loop. A way to do it is to divide the loop into a finite number of parts and use the Noether's theorem on them and then take the limit. So $L_N = \sum_{n=0}^{N} \frac{1}{2}\rho_l[\dot{\vec{r}}(u_n)]^2 du$ and we get the conserved quantity I_N:

$$I_N = \frac{\partial L}{\partial \dot{\vec{r}}}\frac{dh^s(\vec{r})}{ds}\bigg|_{s=0} = \sum_{n=0}^{N} \frac{\partial L}{\partial \dot{\vec{r}}(u_n)}\frac{dh^s(\vec{r}(u_n))}{ds}\bigg|_{s=0} = \sum_{n=0}^{N}[\rho_l \dot{\vec{r}}(u_n)du]\left[\frac{d\vec{r}(u)}{du}\right]_{u=u_n}.$$

Taking the limit, we get the conserved quantity I as:

$$I = \lim_{N\to\infty} I_N = \int_0^L \rho_l \dot{\vec{r}}(u)\frac{d\vec{r}(u)}{du}du.$$

2. Circular meniscus A vertical glass rod of radius b is extends upward from a large dish of water. The angle between the glass surface and the water surface at the circle of contact is a material constant called α. The height of the water at distance r from the axis of the rod is denoted $h(r)$. This $h(r)$ minimizes the sum

$$\int_b^\infty 2\pi r \; dr \left[\rho g h^2 + \gamma\sqrt{1+h'^2}\right],$$

where ρ is the density of water, g is the acceleration of gravity and the surface tension γ is a material constant.

a) Following the strategy used in the midterm, find $h(r)$. Note that the "Hamiltonian" is no longer independent of r.

b) Can you give a direct mechanical interpretation of the "Hamiltonian" in this geometry or that of the midterm?

c) How does $h(r)$ approach its final value as $r \to \infty$?

Solution:

a) The strategy used in the midterm **doesn't work** for this problem, for a subtle and instructive reason. Thus, we us try to follow the strategy and show where it fails. Denoting the entire integrand as $\mathcal{L}$, we note that $\partial\mathcal{L}/\partial h' = 2\pi r\gamma h'/\sqrt{\ldots}$, so that $h'\partial\mathcal{L}/\partial h' = 2\pi r\gamma h'^2/\sqrt{\ldots}$ Denoting $\mathcal{H} \equiv h'\partial\mathcal{L}/\partial h' - \mathcal{L}$, we infer

$$\mathcal{H} = 2\pi r\left[\gamma h'^2/\sqrt{\ldots} - \rho g h^2 - \gamma\sqrt{1+h'^2}\right]$$

$$\mathcal{H} = 2\pi r \left[\gamma h'^2/\sqrt{\ldots} - \rho g h^2 - \gamma(1+h'^2)/\sqrt{\ldots}\right] = 2\pi r \left[-\rho g h^2 - \gamma/\sqrt{\ldots}\right]$$

From Lagrangian mechanics, we know that $d\mathcal{H}/dr = \partial\mathcal{H}/\partial r$.

Our $\mathcal{H}$ depends explicitly on r only through its prefactor $2\pi r$, so that $\partial\mathcal{H}/\partial r = \mathcal{H}/r$.

This innocent-looking statement that I sketched in class is **incorrect**. To find the problem with it, we must recall the meaning of $\partial\mathcal{H}/\partial r$. This means one differentiates with respect to r with the other variables held fixed. The ☐ statement is true if one keeps h and h' fixed. However, in the equation $d\mathcal{H}/dr = \partial\mathcal{H}/\partial r$ the partial derivative must not keep these variables fixed. Instead, $\mathcal{H}$ must be the Legendre-transformed function of $\mathcal{L}$: it must be expressed as a function of h and $p_h \equiv \partial\mathcal{L}/\partial h'$; *these* variables must be held fixed. When we change r and keep p_h fixed, h' need not stay fixed; indeed it does not. The correct $\partial\mathcal{H}/\partial r$ can readily be found using one of the fundamental properties of Legendre transforms that we derived: Legendre transformation preserves partial derivatives with respect to untransformed variables. In our case that means

$$\left(\frac{d\mathcal{H}}{dr} =\right) \left.\frac{\partial\mathcal{H}}{\partial r}\right|_{h,p_h} = -\left.\frac{\partial\mathcal{L}}{\partial r}\right|_{h,h'} = -2\pi\left[\rho g h^2 + \gamma\sqrt{1+h'^2}\right] \left(\neq \frac{\mathcal{H}}{r}\right).$$

I do not see any way to infer a constant of the motion from this equation. Thus, regrettably, I posed a problem which I don't know how to solve.

b) we didn't find a constant, so we have nothing to interpret. However, one may find a constant on physical grounds. The pressure at a given depth must be independent of r. This pressure may be expressed as $\rho g h$ plus a term involving the curvature. This curvature is proportional to h''. It appears that the equation of motion itself expresses the constancy of the pressure.

c) The equation of motion can be simplified for large r when $h' \to 0$. Then the equilibrium equation is the Helmholz equation, and the height decreases to 0 exponentially with a decay length of order $\sqrt{\gamma/(\rho g)}$.

3. Jiggling sphere A particle of mass m slides without friction inside a spherical shell of radius R. The system is in gravity of accleration g.

a) Find the equations of motion for the spherical co-ordinates ϕ and θ, where θ is the polar angle relative to the downward direction.

b) Find the equations if the center of the sphere has a height $z(t)$ that varies sinusoidally in time: $z(t) = z_0 \cos\omega t$.

c) Suppose the initial motion is circular with $\theta = \theta_0 > 0$. Then z begins to vary sinusoidally with very small amplitude z_0 and with a period $1/(2\pi\omega)$ smaller than the orbital period τ. The variation of z results in a sinusoidal variation of θ with the same frequency ω. Find its amplitude.

Solution:

a) Kinetic energy T is given by

$$T = \frac{1}{2}m({v_\phi}^2 + {v_\theta}^2) = \frac{1}{2}m((r\sin\theta\ \dot\phi)^2 + (r\dot\theta)^2) = \frac{1}{2}mr^2(\sin^2\theta\ \dot\phi^2 + \dot\theta^2)$$

Thus

$$\mathcal{L} = T - V = \frac{1}{2}mr^2(\sin^2\theta\ \dot\phi^2 + \dot\theta^2) + mgr\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = mr^2\sin^2\theta\ \dot\phi \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

So that $p_\phi \equiv \partial\mathcal{L}/\partial\dot\phi = mr^2\sin^2\theta\ \dot\phi$ is constant.

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = mr^2\ \dot\theta \quad ; \quad \frac{\partial\mathcal{L}}{\partial\theta} = mr^2\sin\theta\cos\theta\dot\phi^2 - mgr\sin\theta$$

Thus

$$\ddot\theta = \sin\theta(\cos\theta\dot\phi^2 - g/r) = \sin\theta\left[\cos\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2 - g/r\right]$$

$$\ddot{\theta} = \sin\theta \left[\frac{\cos\theta}{\sin^4\theta} \left(\frac{p_\phi}{mr^2} \right)^2 - \frac{g}{r} \right]$$

b) We have seen that Lagrange's equations are still valid in the time-dependent co-ordinates, provided we express T correctly in terms of θ and ϕ. The motion of the sphere can alter v_θ and v_ϕ. Since the motion is perpendicular to the ϕ direction, v_ϕ is in fact unaltered. However, the z motion does alter v_θ: $v_\theta = r\dot{\theta} + \dot{z}\sin\theta$. Now

$$T = \frac{1}{2}m((r\sin\theta\ \dot{\phi})^2 + (r\dot{\theta} + \dot{z}\sin\theta)^2) = \frac{1}{2}mr^2(\sin^2\theta\ \dot{\phi}^2 + \dot{\theta}^2 + \dot{\tilde{z}}^2\sin^2\theta + 2\dot{\tilde{z}}\dot{\theta}\sin\theta),$$

where $\tilde{z} \equiv z/r$.

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\sin^2\theta\ \dot{\phi} \quad ; \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

so that p_ϕ is conserved as before.

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr^2\ (\dot{\theta} + 2\sin\theta\dot{\tilde{z}}) \quad ; \quad \frac{\partial\mathcal{L}}{\partial\theta} = mr^2\sin\theta\cos\theta(\dot{\phi}^2 + \dot{\tilde{z}}^2) + 2\dot{\tilde{z}}\dot{\theta}\cos\theta - mgr\sin\theta$$

Thus

$$\frac{d}{dt}mr^2\ (\dot{\theta} + 2\sin\theta\dot{\tilde{z}}) = mr^2\left(\sin\theta\cos\theta(\dot{\phi}^2 + \dot{\tilde{z}}^2) + 2\dot{\tilde{z}}\dot{\theta}\cos\theta\right) - mgr\sin\theta$$

$$(\ddot{\theta} + 2\sin\theta\ \ddot{\tilde{z}} + 2\cos\theta\ \dot{\theta}\ \dot{\tilde{z}}) = \sin\theta\cos\theta(\dot{\phi}^2 + \dot{\tilde{z}}^2) + 2\dot{\tilde{z}}\dot{\theta}\cos\theta - g/r\sin\theta$$

$$(\ddot{\theta} + 2\sin\theta\ \ddot{\tilde{z}} + 2\cos\theta\ \dot{\theta}\ \dot{\tilde{z}}) = \sin\theta\cos\theta\left(\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2 + \dot{\tilde{z}}^2\right) + 2\dot{\tilde{z}}\dot{\theta}\cos\theta - g/r\sin\theta$$

Identical $\dot{\theta}\dot{\tilde{z}}$ terms cancel on the two sides of the equation. (This should be explained, but I don't have an explanation.) We define $A \equiv (p_\theta/mr^2)^2$. Then

$$\ddot{\theta} + 2\sin\theta\ \ddot{\tilde{z}} = \sin\theta\cos\theta\left(\frac{A}{\sin^4\theta} + \dot{\tilde{z}}^2\right) - \frac{g}{r}\sin\theta$$

Where $\tilde{z} = \tilde{z}_0\cos\omega t$.

c) Putting small quantities on the right, the equation of motion takes the form

$$\ddot{\theta} - \left[\sin\theta\cos\theta\frac{A}{\sin^4\theta} - \frac{g}{r}\sin\theta\right] = +\sin\theta\cos\theta\ \dot{\tilde{z}}^2 - 2\sin\theta\ \ddot{\tilde{z}}$$

We are asked to suppose $\theta - \theta_0 = Ae^{i\omega t}$ with θ small. At $\theta = \theta_0$ the term in [...] vanishes, so we can express it as a Taylor expansion: $[...] = B(g/r)(\theta - \theta_0)$. Where B is a constant involving A and θ_0. On the right side. we may replace θ by θ_0. Thus

$$\ddot{\theta} - B\frac{g}{r}(\theta - \theta_0) = +\sin\theta_0\cos\theta_0\ \dot{\tilde{z}}^2 - 2\sin\theta_0\ \ddot{\tilde{z}}$$

The system is then simply a driven undamped oscillator. Off resonance, only the linear term in $\cot\tilde{z}$ is impotant. At resonance its response grows without bound, and so any resonant contribution is important. The frequencies coming from $\dot{\tilde{z}}$ and $\dot{\tilde{z}}^2$ are ω, 2ω, and 0. Taking account of the variation of $\sin\theta$ and $\cos\theta$ terms on the right, higher multiples of ω are also present.