Physics 316, Problem Set 7, due Tuesday November 15 in class

1. linear independence of basic exterior forms *cf.* p. 167 problem 6. If a vector space has a basis $\mathbf{e}_1 ... \mathbf{e}_n$ then any vector $\boldsymbol{\xi}$ can be represented as $\boldsymbol{\xi} = x_1\mathbf{e}_1 + x_2\mathbf{e}_e + ... + x_n\mathbf{e}_n$. The co-ordinate x_1 of $\boldsymbol{\xi}$ is evidently a 1-form in $\boldsymbol{\xi}$ space, as are all the other co-ordinates. Exterior products of these 1-forms are the 2-forms $x_i \wedge x_j$. There are n^2 2-forms of this type, but only the $n(n-1)/2$ for which $j < i$ are really distinct (since $x_j \wedge x_i = -x_i \wedge x_j$). The question arises whether these $n(n-1)/2$ forms are really distinct themselves.

a) Which of the $x_1 \wedge x_2(\mathbf{e}_i, \mathbf{e}_j)$ are nonzero for the various $i < j$?

b) Are the $x_i \wedge x_j$ with $i < j$ all linearly independent? Why or why not?

Solution:

a) Each $\mathbf{e}_i$ has $x_i = 1$ and $x_j = 0$ for $j \neq i$. So, the $\mathbf{e}$'s that give a nonzero $x_1 \wedge x_2$ are $\mathbf{e}_1$ and $\mathbf{e}_2$.

b) If $x_i \wedge x_j$ for $i < j$ are linearly dependent, we can form a linear combination of them that vanishes for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$: That is for some a_{ij}, and all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$.

$$0 = \omega^2(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \equiv \sum_{i<j} a_{ij} x_i \wedge x_j(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$$

We can clearly find *some* $\boldsymbol{\xi}$'s for which this sum would vanish. The question is whether we can make it vanish for the whole space of $\boldsymbol{\xi}$'s. It must suffice to consider a basis in $\boldsymbol{\xi}$-space. So we consider the $\mathbf{e}$ basis we were given. We consider $x_i \wedge x_j(\mathbf{e}_k, \mathbf{e}_l)$ for $i < j$ and $k < l$. From part a) this vanishes unless $k = i$, and $l = j$. Evidently $\omega^2(\mathbf{e}_1, \mathbf{e}_2) = a_{12} x_1(\mathbf{e}_1) \wedge x_2\mathbf{e}_2$—all the other terms vanish. Therefore we must have $a_{12} = 0$. This argument applies in turn to any $\mathbf{e}_k$ and $\mathbf{e}_l$ with $k < l$. Thus we have all the $a_{ij} = 0$. There is no linear combination of the $x_i \wedge x_j$ that can give the zero 2-form. Thus these $x_i \wedge x_j$ are linearly independent.

2. volumes are linear Given a vector space of n-dimensional vectors $\boldsymbol{\xi}$, we define $A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ to be the area subtended by the vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. It is the area of the parallelogram whose adjacent edges are $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. The text relates this area to a 2-form. Here we investigate the properties of area without using properties of 2-forms. Our vector space has no norm or inner product. Nevertheless we can postulate properties that any area function must satisfy. More generally, we can postulate properties of the volume subtended by 3 vectors or by more vectors. We'll denote all these by $A(\boldsymbol{\xi}_1 .... \boldsymbol{\xi}_k)$. We postulate

i) $A(\lambda\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 ... \boldsymbol{\xi}_k) = \lambda A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 ... \boldsymbol{\xi}_k)$, for any λ, and likewise for the other $\boldsymbol{\xi}$'s. That is, extending one of the vectors by a factor λ necessarily multiplies the area or volume by the same factor.

ii) $(k = 2)$ $A(\boldsymbol{\xi}_1 + \lambda\boldsymbol{\xi}_2, \boldsymbol{\xi}_2) = A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$. I. e. adding some of the second vector to the first has no effect on the area. The addition leaves the base and height of the parallelogram unaffected, hence no change in area. Note that this property does not require us to define perpendicularity.

ii) (general k) For general k adding to any $\boldsymbol{\xi}$ some multiple of another $\boldsymbol{\xi}$ doesn't change A: $A(\boldsymbol{\xi}_1, ...(\boldsymbol{\xi}_i + \lambda\boldsymbol{\xi}_j), ...\boldsymbol{\xi}_k) = A(\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_i, ..., \boldsymbol{\xi}_k)$

a) Using properties i) and ii) with $k = 2$, find $A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1)$. Does this agree with your intuitive notion of area?

b) For general $k \leq n$ suppose $\boldsymbol{\xi}_k = \sum_{i=1}^{k-1} x_i\boldsymbol{\xi}_i$, so the $\boldsymbol{\xi}_1, ...\boldsymbol{\xi}_k$ are linearly dependent. Find $A(\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_k))$.

c) Express $A(\boldsymbol{\xi}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_3, ...\boldsymbol{\xi}_k)$ in terms of $A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, ..., \boldsymbol{\xi}_k)$.

d) Is $A(\boldsymbol{\xi}_1, ...\boldsymbol{\xi}_n)$ linear in $\boldsymbol{\xi}_1$? Why or why not? Evidently, the essential properties of subtended volume lead us to infer the defining properties of an n-form. For $k < n$, linearity can also be proven if we postulate that A behaves as a *projected* area.

Solution:

a) We may subtract from the second $\boldsymbol{\xi}$ any multiple of the first without changing A. Thus,

$$A(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1) = A(\boldsymbol{\xi}_1, 0), = -1A(\boldsymbol{\xi}_1, -0) = -A(\boldsymbol{\xi}_1, 0) = 0$$

b)

$$A(\boldsymbol{\xi}_1, ...\boldsymbol{\xi}_k) = A(x_1, ..., \boldsymbol{\xi}_{k-1}, (x_1\boldsymbol{\xi}_1 + x_2\boldsymbol{\xi}_2 + ... + x_{k-1}\boldsymbol{\xi}_{k-1}))$$

$$= A(\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_{k-1}, (x_2\boldsymbol{\xi}_2 + ... + x_{k-1}\boldsymbol{\xi}_{k-1}) = A(\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_{k-1}, (x_3\boldsymbol{\xi}_3 + ... + x_{k-1}\boldsymbol{\xi}_{k-1})$$

$$= ... = A(\boldsymbol{\xi}_1, ...\boldsymbol{\xi}_{k-1}, 0) = -A(\boldsymbol{\xi}_1, ...\boldsymbol{\xi}_{k-1}, 0) = 0$$

c)

$$A(\mathbf{x},\mathbf{y}) = A(\mathbf{x}+\mathbf{y},\mathbf{y}) = A(\mathbf{x}+\mathbf{y},\mathbf{y}-(\mathbf{x}+\mathbf{y})) = A(\mathbf{x}+\mathbf{y},-\mathbf{x}) = A(\mathbf{y},-\mathbf{x}) = -A(\mathbf{y},\mathbf{x})$$

d) To show linearity, we must show $A(\mathbf{x}+\mathbf{y},\boldsymbol{\xi}_2,...\boldsymbol{\xi}_n) = A(\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_n) + A(\mathbf{y},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_n)$. Necessarily $\mathbf{y}$ is linearly dependent on $\mathbf{x}$ and the $\boldsymbol{\xi}$'s, *i.e.,* $\mathbf{y} = a\mathbf{x} + \sum_i a_i\boldsymbol{\xi}_i$ then

$$A(\mathbf{x}+\mathbf{y},\boldsymbol{\xi}_2,...\boldsymbol{\xi}_k) = (1+a)A(\mathbf{x},\boldsymbol{\xi}_2,...\boldsymbol{\xi}_k) = A(\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k) + aA(\mathbf{x},\boldsymbol{\xi}_2,...\boldsymbol{\xi}_k)$$

$$= A(\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k) + A(a\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k) = A(\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k) + A(a\mathbf{x}+(\sum a_i\boldsymbol{\xi}_i),\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k)$$

$$= A(\mathbf{x},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k) + A(\mathbf{y},\boldsymbol{\xi}_2,...,\boldsymbol{\xi}_k)$$

so that A is indeed linear. The same reasoning shows that it is linear in all its arguments.