Physics 316, Problem Set 8, due Tuesday November 22 in class

1. exercise on cells and chains Suppose one needs to integrate over the unit square with a small square omitted from the middle, like the middle cell of a tic-tac-toe board. This is not a convex polygon, and so it is not a cell. But according to Arnold any integral can be expressed as the integral over a chain. To have a well-defined integral, I must give you an oriented basis. Accordingly, the oriented basis vectors are declared to be, in order, (left to right, down to up).

a) Give the information needed to specify the chain, that is, properties of its polygons, their orientations and their multiplicities. (One is also supposed to specify a map from the co-ordinate space to the manifold, but here the manifold is identical to the co-ordinate space.)

b) In similar fashion, give the information needed to specify the chain that is the boundary of the region in a).

Solution:

a) I take the chain to consist of four rectangles, 1) one of height 1 and base 1/3 with one corner at the origin. 2) an identical one on the right side of the full square, 3) a square of side 1/3 under the omitted square, and 4) a square of side 1/3 above the omitted square. Our space has an oriented basis: horizontal unit vector, vertical unit vector. We keep this same ordering for each of the sub-rectangles. The multiplicities of all these cells is 1, since we want to cover each area exactly once. We could also take a chain that has 2 cells, the first being the entire square, and the second a square of size 1/3 in the middle of the first, but with multiplicity -1.

b) in the first scheme, for each square or rectangle , the boundary consists of four cells: the bottom, right, top and left edges. Each cell must be given an orientation, ie. a direction to be considered the positive direction. We determine the orientation of the bottom edge. The normal vector points down. the normal vector, followed by the positive edge vector are supposed to be oriented the same as the original basis: right, up. If the orientation is OK, I can deform (down, forward) into (right, up) smoothly without making my vectors linearly dependent along the way. That means the determinant can't change sign. For example, we can rotate (right, up) into (down, right), so (down, right) is an oriented basis. The forward or positive direction along the boundary is to the right. By similar reasoning, the right side has a positive direction that is upward. One may also note that the positive direction cannot reverse at the boundary of two adjacent cells. Since the two boundary bases have the same orientation as the body, they must have the same orientation as each other. That means one must be able to deform the (normal, forward) of one cell smoothly into (normal, forward) for the adjacent one. If the forward direction reversed, this would not work.

2. Pyramid of Cheops (*cf.* Glossary entry on orientation.) The Pyramid of Cheops in Egypt has a square base and four triangular sides. One of these sides faces east, so a horizontal line on it points north and south. I declare the following to be an oriented basis on the earth: (eastward, northward, upward). I wish to take as basis for the east face either (northward, slanting upward) or (northward, slanting down).

a) Which of these bases is oriented according to Arnold's convention? Why? Please plod through the Arnold reasoning; don't just use the right hand rule. There is no right hand rule in higher dimensions—not with human hands anyway.

b) What should be an oriented basis for the north face? the bottom face?

c) Specify the chain for the boundary of the east face.

d) Specify the chain for the boundary of all the faces, ie. the boundary of the boundary of the Pyramid. Justify for this example Arnold's claim that this boundary vanishes.

Solution: We try (northward, slanting upward). We make a 3-dimensional basis by adding the normal vector: (out-of-face, northward, upward). We now smoothly deform (eg rotate) this so that out-of-face points upward. The northward vector remains the same and the up-slanting vector points to the west. Our basis is then (upward, northward, westward). Permuting the first and last, we get -(westward, northward, upward). Reversing the first one, we get +(eastward, northward, upward), our declared oriented basis. Since the orientation of our guessed basis is the same as that of our oriented basis, it itself is oriented. So (northward, slanting-up) is the good basis.