Physics 316, Problem Set 9, due Thursday December 1 in class

1. conserved 2-form in accelerator A bunch of charged particles of unit mass enters an accelerator along the z direction. Their mutual interaction is negligible. Upon entry, no forces are acting. There is no special axial or other symmetry. These particles have a spread of x-velocity of Δv_x and a spread of x positions Δx. Upon exit from the accelerator, the bunch is monitored at a point with no forces acting. Those particles which started at the same y, z, v_x, v_y, v_z but differing by Δx now lie in a line along a co-ordinate q, which is a linear combination of x, y, z, v_x, v_y, v_z. Likewise, particles that started with the same x, y, z, v_y, v_z, but differing by Δv_x lie along a direction p which is another linear combination of x, y, z, p_x, p_y, and p_z. Suppose for definiteness that particles separated by 1 mm in the x direction (with identical velocities) end up separated by 1 mm in y and 2 m/sec in v_z, with no separation in x, z, v_x or v_y. Likewise, particles initially separated by 1 m/sec in v_x end up separated by A mm in x, A m/sec in v_x and $3A$ m/sec in v_y, with no separation in the other 3 phase space co-ordinates. (I'm just making up the proportions of the different phase space components.)

a) Suppose that the accelerator is purely electrostatic. The conservation of the symplectic form ω^2 dictates the value of A. What is it?

b) Suppose that the accelerator contains static magnets, so that the forces depend on velocity. Is it still possible to determine the value of A? What is it?

c) Now suppose that the accelerator exerts time-dependent Lorentz forces on the particles. Is it still possible to determine A? Discuss.

d) Suppose the final point had been within the accelerator, where forces were acting. How would your conclusions in a), b) c) change?

Solution: To evaluate ω^2 we need two vectors in the 2n-dimensional tangent space: $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. For $\boldsymbol{\xi}$ we take the initial displacement in the x direction; for $\boldsymbol{\eta}$ we take the initial displacement in the v_x direction. Since v_x is conjugate to x, $\omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = dx \wedge dv_x = dx(\boldsymbol{\xi})dv_x(\boldsymbol{\eta}) - dx(\boldsymbol{\eta})dv_x(\boldsymbol{\xi})$. Now, $dx(\boldsymbol{\xi}) = 1$ mm, $dx(\boldsymbol{\eta}) = 0$,$dv_x(\boldsymbol{\xi}) = 0$, $dv_x(\boldsymbol{\eta}) = 1$m/sec. Thus $\omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = .001$ m^2/sec. We denote the image of $\boldsymbol{\xi}$ at the final time as $\boldsymbol{\xi}'$, and likewise with $\boldsymbol{\eta}'$. We denote the conserved 2-form at the image point as $\omega^{2\prime}$. Then the conservation of ω^2 means $\omega^{2\prime}(\boldsymbol{\xi}', \boldsymbol{\eta}') = \omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = .001$ m^2/sec.

a) At the final point, we may choose coordinates x, y, z; since no forces are acting, the conjugate momenta are v_x, v_y and v_z. The non-vanishing contributions in $\omega^{2\prime}(\boldsymbol{\xi}', \boldsymbol{\eta}')$ give

$$\omega^{2\prime}(\boldsymbol{\xi}', \boldsymbol{\eta}') = dy \wedge dv_y(\boldsymbol{\xi}', \boldsymbol{\eta}') = dy(\boldsymbol{\xi}')dv_y(\boldsymbol{\eta}') - dy(\boldsymbol{\eta}')dv_y\boldsymbol{\xi}') = .003A\text{m}^2/\text{sec}$$

Comparing, we infer $.001 = .003A$ or $A = 1/3$

b), c) The reasoning in a) didn't depend on the nature of the Hamiltonian flow between the initial and final points. Hence the value of A remains unchanged.

d) If the particle is in the middle of the accelerator where forces act, it is not generally true that v_x is the momentum conjugate to x, etc. This is true so long as the system has a conventional kinetic energy and a potential energy that depends only on position. This this is the case in a) but not b) or c). Thus we can't determine A for cases b) or c) if the final point is inside the accelerator where velocity dependent forces act.

2. Canonical transformation problem: Lenz vector A planet of unit mass revolves in the x-y plane about a heavy star. The Hamiltonian is given by $\mathcal{H} = \frac{1}{2}v^2 - k/r$.

a) Find the vector field in phase space (x, y, v_x, v_y) generated by the scalar field x. Arnold's term for this vector field is Idx. That is, express Idx as a linear combination of the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{v}}_x$, and $\hat{\mathbf{v}}_y$.

b) Find the vector field in phase space generated by the scalar field $x(v^2)$, *i.e.,* $I(xv^2)$

c) Find the vector field in phase space generated by the scalar field $v_x(v \cdot r)$.

d) Use Poisson Brackets to find the time derivitives of x, xv^2 and $v_x(v \cdot r)$.

e) Find the a linear combination of x, xv^2 and $v_x(v \cdot r)$ that has a vanishing time derivitive for a planet with energy E. This constant is generally denoted A_x; it is the x component of the famous Lenz vector.

f) *Optional:* Transform a circular orbit of energy E using the canonical transformation generated by A_x. What happens to the orbit? Is it still a Kepler orbit? What happens to its A_y?, its angular momentum L?

Solution:

a) The vector $\boldsymbol{\xi}(x, y, v_x, v_y) = Idx$ is determined by

$$\omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = dx(\boldsymbol{\eta})$$

for all $\boldsymbol{\eta}$. or

$$(dx \wedge dv_x + dy \wedge dv_y)(\boldsymbol{\xi}, \boldsymbol{\eta}) = dx(\boldsymbol{\eta})$$

or

$$dx(\boldsymbol{\xi})dv_x(\boldsymbol{\eta}) - dx(\boldsymbol{\eta})dv_x(\boldsymbol{\xi}) + (x \to y) = dx(\boldsymbol{\eta})$$

Evidently we could have $\boldsymbol{\xi} = -\hat{\mathbf{v}}_x$, Then $dv_x(\xi) = -1$ and the $dy \wedge dv_y$ would be zero. Since this is an invertible linear problem, the solution is unique. Idx is a unit translation in the v_x direction. Oddly, the vector Idx has dimensions different from those of x; thus the magnitude of Idx is altered if we change our unit of measurement. This ambiguity is an inevitable consequence of the lack of a notion of length in phase space.

b)

$$d(xv^2) = v^2 dx + 2xv_x dv_x + 2xv_y dv_y$$

$$dx(\boldsymbol{\xi})dv_x(\boldsymbol{\eta}) - dx(\boldsymbol{\eta})dv_x(\boldsymbol{\xi}) + (x \to y) = \left(v^2 dx + 2xv_x dv_x + 2xv_y dv_y\right)(\boldsymbol{\eta})$$

looking at the x part, we have

$$dx(\boldsymbol{\xi})dv_x(\boldsymbol{\eta}) - dx(\boldsymbol{\eta})dv_x(\boldsymbol{\xi}) = v^2 dx(\boldsymbol{\eta}) + 2xv_x dv_x(\boldsymbol{\eta})$$

Thus $dx(\boldsymbol{\xi}) = 2xv_x$, and $dv_x(\boldsymbol{\xi}) = v^2$. Using the dy and dv_y part, $dy(\boldsymbol{\xi}) = 2xv_y$. Thus

$$Id(xv^2) = 2xv_x\hat{\mathbf{x}} + 2xv_y\hat{\mathbf{y}} + v^2\hat{\mathbf{v}}_x = 2x\mathbf{v} + v^2\hat{\mathbf{v}}_x$$

c)

$$d(v_x(v \cdot r)) = (v \cdot r)dv_x + v_x\ r \cdot dv + v_x(v \cdot dr)$$

$$dx(\boldsymbol{\xi})dv_x(\boldsymbol{\eta}) - dx(\boldsymbol{\eta})dv_x(\boldsymbol{\xi}) + (x \to y) = (v \cdot r)dv_x(\boldsymbol{\eta}) + v_x\ r \cdot dv(\boldsymbol{\eta}) + v_x(v \cdot dr(\boldsymbol{\eta}))$$

From which $dx(\boldsymbol{\xi}) = (v \cdot r) + v_x\ x$, $dy(\boldsymbol{\xi}) = v_x\ y$, $dv_x(\boldsymbol{\xi}) = -{v_x}^2$; $dv_y(\boldsymbol{\xi}) = -v_y\ v_x$ So that

$$Id(v_x(r \cdot v)) = [(v \cdot r) + x\ v_x]\hat{\mathbf{x}} + y\ v_x\hat{\mathbf{y}} - {v_x}^2\hat{\mathbf{v}}_x - \hat{\mathbf{v}}_y v_y v_x$$

$$= (v \cdot r)\hat{\mathbf{x}} + v_x(\mathbf{r} + \mathbf{v})$$

d)

$$\{x, \mathcal{H}\} = \partial\mathcal{H}/\partial v_x = v_x$$

$$\{xv^2, \mathcal{H}\} = \frac{\partial xv^2}{\partial x}\frac{\partial \mathcal{H}}{\partial v_x} - \frac{\partial xv^2}{\partial v_x}\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial xv^2}{\partial y}\frac{\partial \mathcal{H}}{\partial v_y} - \frac{\partial xv^2}{\partial v_y}\frac{\partial \mathcal{H}}{\partial y}$$

$$= v^2 v_x - 2x\ v_x(kx/r^3) + 0 - 2x\ v_y(ky/r^3)$$

$$= v^2 v_x - 2(xv_x + yv_y)kx/r^3 = v^2 v_x - 2(v \cdot r)kx/r^3$$

$$\{v_x(v \cdot r), \mathcal{H}\} = \frac{\partial v_x(v \cdot r)}{\partial x}\frac{\partial \mathcal{H}}{\partial v_x} - \frac{\partial v_x(v \cdot r)}{\partial v_x}\frac{\partial \mathcal{H}}{\partial x} + \frac{\partial v_x(v \cdot r)}{\partial y}\frac{\partial \mathcal{H}}{\partial v_y} - \frac{\partial v_x(v \cdot r)}{\partial v_y}\frac{\partial \mathcal{H}}{\partial y}$$

$$= v_x^2\ v_x - [(v \cdot r) + v_x x]kx/r^3 + v_x v_y v_y - v_x y\ ky/r^3$$

$$= v_x\ v^2 - kv_x/r - (v \cdot r)kx/r^3$$

e) We may eliminate the $(v \cdot r)$ terms by combining $2\{v_x(v \cdot r), \mathcal{H}\} - \{xv^2, \mathcal{H}\}$:

$$2\{v_x(v \cdot r), \mathcal{H}\} - \{xv^2, \mathcal{H}\} = 2v_x(v^2 - k/r) - v^2 v_x = 2v_x(\frac{1}{2}v^2 - k/r) = 2v_x E = 2\{x, \mathcal{H}\}E$$

Thus $2v_x(v \cdot r) - xv^2 - 2xE$ is the desired constant of the motion, and we denote it by A_x.