Solution to Problem Set 4

Solution:

a) Express the kinetic energy T in terms of the given coordinates:

$$T_{cm} = \frac{1}{2} \cdot 2m \cdot (\dot{r}^2 + r^2\dot{\phi}^2), \tag{1}$$

$$\begin{aligned} T &= T_{cm} + \frac{1}{2}\sum_i m_i r_i^2 \omega_i^2 \\ &= m(\dot{r}^2 + r^2\dot{\phi}^2) + mL^2\left(\dot{\theta} + \dot{\phi}\right)^2. \end{aligned} \tag{2}$$

b) The Lagrangian of the system:

$$L = T - U, \tag{3}$$

$$U = \frac{-GMm}{\sqrt{r^2 + L^2 + 2rL\cos\theta}} + \frac{-GMm}{\sqrt{r^2 + L^2 - 2rL\cos\theta}}, \tag{4}$$

where M is the sun's mass.

c) Since $\partial L/\partial\phi = 0$, p_ϕ is the constant of motion:

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = 2mr^2\dot{\phi} + 2mL^2\left(\dot{\phi} + \dot{\theta}\right) = const. \tag{5}$$

d) Using Euler-Lagrange equation, it's straightforward to write down those equations of motion:

$$\text{for } r: \qquad 2\ddot{r} = 2r\dot{\phi}^2 - GM\left(\frac{r + L\cos\theta}{(r^2 + L^2 + 2rL\cos\theta)^{3/2}} + \frac{r - L\cos\theta}{(r^2 + L^2 - 2rL\cos\theta)^{3/2}}\right), \tag{6}$$

$$\text{for } \phi: \qquad 2r\dot{r}\dot{\phi} + \left(r^2 + L^2\right)\ddot{\phi} = -L^2\ddot{\theta}, \tag{7}$$

$$\text{for } \theta: \quad 2L^2\ddot{\theta} = -2L^2\ddot{\phi} - GMrL\sin\theta\left(\frac{1}{(r^2 + L^2 + 2rL\cos\theta)^{3/2}} - \frac{1}{(r^2 + L^2 - 2rL\cos\theta)^{3/2}}\right). \tag{8}$$

e) Find the Hamiltonian of the system and Hamiltonian's equations of motion:

$$p_r = \frac{\partial L}{\partial\dot{r}} = 2m\dot{r}, \tag{9}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = 2mr^2\dot{\phi} + 2mL^2\left(\dot{\phi} + \dot{\theta}\right), \tag{10}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = 2mL^2\left(\dot{\theta} + \dot{\phi}\right), \tag{11}$$

and therefore,

$$H = \sum_i p_i \dot{q}_i - L$$
$$= \frac{p_r^2}{4m} + \frac{(p_\phi - p_\theta)^2}{4mr^2} + \frac{p_\theta^2}{4mL^2} + \frac{-GMm}{\sqrt{r^2 + L^2 + 2rL\cos\theta}} + \frac{-GMm}{\sqrt{r^2 + L^2 - 2rL\cos\theta}}. \tag{12}$$

Hamiltonian's equations of motion:

$$\dot{p}_i = \frac{\partial H}{\partial q_i}, \quad \dot{q}_i = -\frac{\partial H}{\partial p_i}, \tag{13}$$

thus:

$$\dot{r} = \frac{p_r}{2m}, \tag{14}$$

$$\dot{p}_r = \frac{(p_\theta - p_\phi)^2}{2mr^3} - GMm\left(\frac{r + L\cos\theta}{(r^2 + L^2 + 2rL\cos\theta)^{3/2}} + \frac{r - L\cos\theta}{(r^2 + L^2 - 2rL\cos\theta)^{3/2}}\right); \tag{15}$$

$$\dot{\phi} = \frac{p_\phi - p_\theta}{2mr^2}, \tag{16}$$

$$\dot{p}_\phi = 0; \tag{17}$$

$$\dot{\theta} = \frac{p_\phi - p_\theta}{2mr^2} + \frac{p_\theta}{2mL^2}, \tag{18}$$

$$\dot{p}_\theta = GMmrL\sin\theta\left(\frac{1}{(r^2 + L^2 + 2rL\cos\theta)^{3/2}} - \frac{1}{(r^2 + L^2 - 2rL\cos\theta)^{3/2}}\right). \tag{19}$$

f) We need the initial conditions $\left(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}\right)_{t=0}$ to solve the dynamical system

$$r_0 = R - L, \ \theta_0 = \phi_0 = 0, \tag{20}$$

$$\dot{r}_{t=0} = 0. \tag{21}$$

According to the previous discussions, p_ϕ [Eq. (5)] is the total angular momentum of the system, which is conserved. We have

$$p_\phi = 2m\left(R - L\right)^2 \dot{\phi}_{t=0} + 2mL^2\left(\dot{\phi}_{t=0} + \dot{\theta}_{t=0}\right) = m\left(V_e R + V_v\left(R - 2L\right)\right). \tag{22}$$

Additionally, according to our coordinate system:

$$V_v - V_e = 2L(\dot{\theta}_{t=0} + \dot{\phi}_{t=0}), \tag{23}$$

where

$$V_v = \sqrt{\frac{GM}{R - 2L}}, V_e = \sqrt{\frac{GM}{R}}. \tag{24}$$

Therefore using Eqs. (22) and (23), we find

$$\dot{\phi}_{t=0} = \frac{V_e\left(R + L\right) + V_e\left(R - 3L\right)}{2\left(R - L\right)^2}, \quad (?? \text{ —TW}) \tag{25}$$

$$\dot{\phi}_{t=0} = (V_e + V_v)/(2r_0) \quad (\text{—TW}) \tag{26}$$

$$\dot{\theta}_{t=0} = \frac{V_v - V_e}{2L} - \frac{V_e\,(R+L) + V_e\,(R-3L)}{2\,(R-L)^2}. \quad (\text{?? —TW}) \tag{27}$$

$$\dot{\theta}_{t=0} = -\dot{\phi}_0 + \frac{1}{2}(V_v - V_e)/L \quad (\text{—TW}) \tag{28}$$

g) —see next page (—TW)

solution to problem set 4 part g

The question asks you to discuss whether the motion is confined to small oscillations of θ or whether the bridge rotates and θ increases indefinitely. I wanted you to try to formulate a strategy like the one below. Here I work through such a strategy in great detail to show it is do-able. I did not expect you to work through this. I include the details only for those who wish to verify the steps.

We denote the conserved momentum $p_\phi \equiv -B$. In order to decide if θ is confined in a subset of its full range or not, we must characterize the phase portrait of the motion, as one does for a simple pendulum. In that simple case, θ is restricted in range when the energy is near to its minimum value of stable equilibrium. For our system there can also be equilibrium points or fixed points in phase space. Our phase space consists of (r, θ) and their conjugate momenta; we treat B as a parameter and ignore ϕ.

Our reasoning is as outlined in class. Given B we expect that there is a point in phase space with lowest energy $H = \bar{H}$. Points with energy near this minimum H must be confined near the equilibrium point. In particular θ must be confined near its equilibrium value. Conversely, by giving the system a sufficiently large initial $\dot{\theta}$, we can surely make θ increase indefinitely instead of oscillating. If we reduce H from this rotating state sufficiently, we must cross a threshold where θ is confined to oscillation. At this point, it seems clear that the transit time from the initial θ to some given large θ (*eg.* 16π) must diverge. If this time is to diverge, the time derivitives of phase space co-ordinates must go to zero. That is, there must be a point of (unstable) equilibrium along the marginally escaping orbit. We denote its energy by $\hat{H}$. Evidently if $H < \hat{H}$, no escape is possible, and θ must oscillate. On the other hand, if $H > \hat{H}$, it is plausible to suppose that the θ increases without bound. An aim of the course is to lay the groundwork to prove these suppositions.

We first identify fixed points, where $\dot{r}, \dot{\theta}, \dot{p}_r$ and $\dot{p}_\phi$ vanish. From the condition $\dot{p}_\theta = 0$ (eq. 19 above) we infer $\sin\theta = 0$ or the large expression $(..) = 0$. The first choice implies $\theta = 0, \pi$. The second choice implies $\cos\theta = 0$, or $\theta = \pm\pi/2$. We analyze the two cases in turn

$\theta = 0$ fixed point Evidently from the $\dot{r} = 0$ condition, we infer $p_r = 0$. We tentatively identify this as the stable equilibrium point and denote it by $\bar{r}, \bar{\theta}, \bar{p}_\theta, \bar{H}$, etc.. Now the $\dot{\theta} = 0$ condition above yields

$$0 = \dot{\theta} = \frac{p_\phi - \bar{p}_\theta}{2m\bar{r}^2} + \frac{\bar{p}_\theta}{2mL^2}$$

We define $L/r \equiv \lambda$ and note that $0 < \lambda < 1$. thus so that

$$p_\phi - \bar{p}_\theta = -\frac{\bar{p}_\theta}{\bar{\lambda}^2}; \quad B = \bar{p}_\theta(\bar{\lambda}^{-2} - 1)$$

Next, we deal with the $\dot{p}_r$ condition, using $p_\phi - \bar{p}_\theta = -\bar{p}_\theta\ \bar{\lambda}^{-2}$:

$$0 = \dot{p}_r = \frac{\left(\bar{p}_\theta\bar{\lambda}^{-2}\right)^2}{2m\bar{r}^3} - GMm\left(\frac{\bar{r}+L}{\left(\bar{r}^2+L^2+2\bar{r}L\right)^{3/2}} + \frac{\bar{r}-L}{\left(\bar{r}^2+L^2-2\bar{r}L\right)^{3/2}}\right)$$

$$0 = \frac{\left(\bar{p}_\theta\bar{\lambda}^{-2}\right)^2}{2m\bar{r}^3} - \frac{GMm}{\bar{r}^2}\left(\frac{1+\bar{\lambda}}{\left(1+\bar{\lambda}^2+2\bar{\lambda}\right)^{3/2}} + \frac{1-\bar{\lambda}}{\left(1+\bar{\lambda}^2-2\bar{\lambda}\right)^{3/2}}\right)$$

$$0 = \frac{1}{2}\bar{p}_\theta^2\bar{\lambda}^{-4} - GMm^2\bar{r}\left(\frac{1+\bar{\lambda}}{\left(1+\bar{\lambda}^2+2\bar{\lambda}\right)^{3/2}} + \frac{1-\bar{\lambda}}{\left(1+\bar{\lambda}^2-2\bar{\lambda}\right)^{3/2}}\right)$$

$$0 = \frac{1}{2}\bar{p}_\theta^2\bar{\lambda}^{-4} - GMm^2\bar{r}\left(\frac{1}{\left(1+\bar{\lambda}\right)^2} + \frac{1}{\left(1-\bar{\lambda}\right)^2}\right)$$

$$\frac{1}{2}\bar{p}_\theta{}^2\bar{\lambda}^{-4} = GMm^2\bar{r}\left(\frac{\left(1-\bar{\lambda}\right)^2+\left(1+\bar{\lambda}\right)^2}{\left(1+\bar{\lambda}\right)^2\left(1-\bar{\lambda}\right)^2}\right)$$

$$\frac{1}{2}\bar{p}_\theta{}^2\bar{\lambda}^{-4} = GMm^2\bar{r}\left(\frac{2+2\bar{\lambda}^2}{\left(1-\bar{\lambda}^2\right)^2}\right)$$

$$\bar{p}_\theta{}^2 = 4\ GMm^2\bar{r}\ \bar{\lambda}^4\left(\frac{1+\bar{\lambda}^2}{\left(1-\bar{\lambda}^2\right)^2}\right)$$

We may use these equilibrium conditions to determine B and $\bar{p}_\theta$ as functions of the given $\bar{r}$. We could invert them to obtain $\bar{r}$ and $\bar{p}_\theta$ in terms of given B. This involves transcendental equations, so we'll postpone it.

We now determine the equilibrium energy $\bar{H}$ using $p_r = 0$ and $p_\phi - \bar{p}_\theta = -\bar{p}_\theta/\bar{\lambda}^2$

$$\bar{H} = \frac{\left(\bar{p}_\theta/\bar{\lambda}^2\right)^2}{4m\bar{r}^2} + \frac{\bar{p}_\theta^2}{4mL^2} + \frac{-GMm/\bar{r}}{\sqrt{1+\bar{\lambda}^2+2\bar{\lambda}}} + \frac{-GMm/\bar{r}}{\sqrt{1+\bar{\lambda}^2-2\bar{\lambda}}}.$$

$$\bar{H} = \frac{\bar{p}_\theta{}^2/\bar{\lambda}^4}{4m\bar{r}^2} + \frac{\bar{p}_\theta^2}{4mL^2} - \frac{GMm}{\bar{r}}\left(\frac{1}{1+\bar{\lambda}} + \frac{1}{1-\bar{\lambda}}\right).$$

$$\bar{H} = \frac{\bar{p}_\theta{}^2}{4m\bar{r}^2}\left(\frac{1}{\bar{\lambda}^4} + \frac{1}{\bar{\lambda}^2}\right) - \frac{GMm}{\bar{r}}\left(\frac{1}{1+\bar{\lambda}} + \frac{1}{1-\bar{\lambda}}\right).$$

Using our expression for $\bar{p}_\theta{}^2$,

$$\bar{H} = \frac{GMm}{\bar{r}}\left(\frac{1+\bar{\lambda}^2}{1-\bar{\lambda}^2}\right)^2 - \frac{GMm}{\bar{r}}\left(\frac{1}{1+\bar{\lambda}} + \frac{1}{1-\bar{\lambda}}\right).$$

$$\bar{H} = \frac{GMm}{\bar{r}}\left[\left(\frac{1+\bar{\lambda}^2}{1-\bar{\lambda}^2}\right)^2 - \left(\frac{1}{1+\bar{\lambda}} + \frac{1}{1-\bar{\lambda}}\right)\right].$$

$$\bar{H} = \frac{GMm}{\bar{r}}\left[\frac{(1+\bar{\lambda}^2)^2}{(1-\bar{\lambda}^2)^2} - \left(\frac{2}{1-\bar{\lambda}^2}\right)\right].$$

$$\bar{H} = \frac{GMm}{\bar{r}}\left[\frac{(1+\bar{\lambda}^2)^2}{(1-\bar{\lambda}^2)^2} - \frac{2(1-\bar{\lambda}^2)}{(1-\bar{\lambda}^2)^2}\right].$$

$$\bar{H} = \frac{GMm}{\bar{r}}\left[\frac{(1+\bar{\lambda}^2)^2 - 2(1-\bar{\lambda}^2)}{(1-\bar{\lambda}^2)^2}\right].$$

$$\bar{H} = \frac{GMm}{\bar{r}}\left[\frac{-1+4\bar{\lambda}^2+\bar{\lambda}^4}{(1-\bar{\lambda}^2)^2}\right].$$

$\theta = \pi/2$ fixed point We tentatively identify this one as the unstable equilibrium point and denote it by $\hat{r}, \hat{H}$, etc. As with the $\theta = 0$ fixed point, we again find $\hat{p}_r = 0$, and $p_\phi - \hat{p}_\theta = -\hat{p}_\theta\ \hat{\lambda}^{-2}$. To find a final constraint relating $\hat{p}_\theta$ to $\hat{r}$ we write

$$0 = \dot{\hat{p}}_r = \frac{(\hat{p}_\theta - p_\phi)^2}{2m\hat{r}^3} - GMm\left(\frac{\hat{r}+L\cos\theta}{(\hat{r}^2+L^2+2\hat{r}L\cos\theta)^{3/2}} + \frac{\hat{r}-L\cos\theta}{(\hat{r}^2+L^2-2\hat{r}L\cos\theta)^{3/2}}\right)$$

Using $\cos\theta = 0$, $p_\phi - \hat{p}_\theta = -\hat{p}_\theta\ \hat{\lambda}^{-2}$, and $L/\hat{r} = \hat{\lambda}$, this becomes

$$0 = \frac{(\hat{p}_\theta\ \hat{\lambda}^{-2})^2}{2m\hat{r}^3} - GMm\left(\frac{\hat{r}}{(\hat{r}^2+L^2)^{3/2}} + \frac{\hat{r}}{(\hat{r}^2+L^2)^{3/2}}\right)$$

$$0 = \frac{(\hat{p}_\theta\ \hat{\lambda}^{-2})^2}{2m\hat{r}^3} - \frac{GMm}{\hat{r}^2}\frac{2}{\left(1+\hat{\lambda}^2\right)^{3/2}}$$

So that

$$\hat{p_\theta}^2 = GMm^2\hat{r}\frac{4\hat{\lambda}^4}{\left(1+\hat{\lambda}^2\right)^{3/2}}$$

Now we obtain $\hat{H}$:

$$\hat{H} = \frac{\left(\hat{p}_\theta\ \hat{\lambda}^{-2}\right)^2}{4m\hat{r}^2} + \frac{\hat{p}_\theta^2}{4mL^2} + \frac{-GMm}{\sqrt{\hat{r}^2+L^2}} + \frac{-GMm}{\sqrt{\hat{r}^2+L^2}}$$

$$\hat{H} = \frac{\hat{p_\theta}^2\ \hat{\lambda}^{-4}}{4m\hat{r}^2} + \frac{\hat{p}_\theta^2}{4m\hat{r}^2\hat{\lambda}^2} + \frac{-2GMm}{\sqrt{\hat{r}^2+L^2}}$$

$$\hat{H} = \frac{\hat{p_\theta}^2}{4m\hat{r}^2}(\hat{\lambda}^{-4}+\hat{\lambda}^{-2}) + \frac{-2GMm/\hat{r}}{\sqrt{1+\hat{\lambda}^2}}$$

$$\hat{H} = \frac{GMm}{\hat{r}}\frac{\hat{\lambda}^4}{\left(1+\hat{\lambda}^2\right)^{3/2}}(\hat{\lambda}^{-4}+\hat{\lambda}^{-2}) + \frac{-2GMm/\hat{r}}{\sqrt{1+\hat{\lambda}^2}}$$

$$\hat{H} = \frac{GMm}{\hat{r}}\left(\frac{1}{\sqrt{1+\hat{\lambda}^2}} + \frac{-2}{\sqrt{1+\hat{\lambda}^2}}\right)$$

$$\hat{H} = \frac{-GMm}{\hat{r}}\frac{1}{\sqrt{1+\hat{\lambda}^2}}$$

Initial H We now compare this $\hat{H}$ to the initial value specified in the problem, which we denote H_0. As told in the solution above,

$$p_\phi = m\left(V_e R + V_v\left(R-2L\right)\right) = m\left(V_e(r_0+L) + V_v\left(r_0-L\right)\right)$$

Where

$$V_v = \sqrt{\frac{GM}{R-2L}} = \sqrt{\frac{GM}{r_0}}\sqrt{\frac{r_0}{r_0-L}} = \sqrt{\frac{GM}{r_0}}\sqrt{\frac{1}{1-\lambda_0}}$$

and

$$V_e = \sqrt{\frac{GM}{r_0}}\sqrt{\frac{1}{1+\lambda_0}}$$

Thus

$$-B = p_\phi = mr_0\sqrt{\frac{GM}{r_0}}\left(\sqrt{\frac{1}{1+\lambda_0}}(1+\lambda_0) + \sqrt{\frac{1}{1-\lambda_0}}\left(1-\lambda_0\right)\right) = \sqrt{GMm^2r_0}\left(\sqrt{1+\lambda_0}+\sqrt{1-\lambda_0}\right)$$

To obtain $p_{\theta 0}$ we use the relation between $p_{\theta 0}$ and $\dot{\theta}$:

$$\dot{\theta_0} = \frac{p_\phi - p_\theta}{2mr_0^2} + \frac{p_{\theta 0}}{2mL^2} = \frac{1}{2mr_0^2}\left(p_\phi - p_{\theta 0} + p_{\theta 0}\lambda_0^{-2}\right) = \frac{-1}{2mr_0^2}\left(B + p_{\theta 0}(1-\lambda_0^{-2})\right)$$

$$\dot{\theta}_0 + \dot{\phi}_0 = \omega = (V_v - V_0)/L = (V_e - \frac{1}{2}(V_e+V_v))/L = \frac{1}{2}(V_v - V_e)/L$$

$$2mr_0\dot{\phi}_0 = mV_e + mV_v; \quad \dot{\phi}_0 = (V_e+V_v)/(2r_0)$$

$$\dot{\theta}_0 = -\dot{\phi}_0 + \frac{1}{2}(V_v - V_e)/L = -(V_e+V_v)/(2r_0) + \frac{1}{2}(V_v - V_e)/L = \frac{-1}{2r_0}\left(V_e(1+1/\lambda_0) + V_v(1-1/\lambda_0)\right)$$

$$\dot{\theta}_0 = \frac{-1}{2r_0}\sqrt{\frac{GM}{r_0}}\left(\sqrt{\frac{1}{1+\lambda_0}}(1+1/\lambda_0) + \sqrt{\frac{1}{1-\lambda_0}}(1-1/\lambda_0)\right)$$

$$\dot{\theta}_0 = \frac{-1}{2r_0\lambda_0}\sqrt{\frac{GM}{r_0}}\left(\sqrt{\frac{1}{1+\lambda_0}}(\lambda_0+1) - \sqrt{\frac{1}{1-\lambda_0}}(1-\lambda_0)\right)$$

$$\dot{\theta}_0 = \frac{-1}{2r_0\lambda_0}\sqrt{\frac{GM}{r_0}}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)$$

Combining these two expressions for $\dot{\theta}_0$, we find

$$\frac{-1}{2mr_0^2}\left(B + p_{\theta 0}(1-\lambda_0^{-2})\right) = \dot{\theta}_0 = \frac{-1}{2r_0\lambda_0}\sqrt{\frac{GM}{r_0}}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)$$

$$B + p_{\theta 0}(1-\lambda_0^{-2}) = \frac{1}{\lambda_0}\sqrt{GMm^2r_0}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)$$

$$p_{\theta 0}(1-\lambda_0^{-2}) = -B + \frac{1}{\lambda_0}\sqrt{GMm^2r_0}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)$$

$$p_{\theta 0}(1-\lambda_0^{-2}) = \sqrt{GMm^2r_0}\left(\sqrt{1+\lambda_0} + \sqrt{1-\lambda_0} + \frac{1}{\lambda_0}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)\right)$$

$$p_{\theta 0} = \sqrt{GMm^2r_0}\frac{-\lambda_0^2}{(1-\lambda_0^2)}\left(\sqrt{1+\lambda_0} + \sqrt{1-\lambda_0} + \frac{1}{\lambda_0}\left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)\right)$$

$$p_{\theta 0} = \sqrt{GMm^2r_0}\frac{-1}{(1-\lambda_0^2)}\left(\lambda_0^2\sqrt{1+\lambda_0} + \lambda_0^2\sqrt{1-\lambda_0} + \lambda_0\sqrt{1+\lambda_0} - \lambda_0\sqrt{1-\lambda_0}\right)$$

The initial kinetic energy T_0 is given by

$$T_0 = mr_0^2(\dot{\phi}_0)^2 + mL^2(\dot{\theta}_0)^2$$

$$T_0 = mr_0^2((V_e+V_v)/(2r_0))^2 + mL^2(\frac{1}{2r_0}(V_e(1+1/\lambda_0) + V_v(1-1/\lambda_0)))^2$$

$$T_0 = mr_0^2\frac{GM}{r_0}\left(\sqrt{\frac{1}{1+\lambda_0}} + \sqrt{\frac{1}{1-\lambda_0}}\right)^2/(2r_0))^2 + mL^2\frac{GM}{r_0}(\frac{1}{2r_0})^2\left(\sqrt{\frac{1}{1+\lambda_0}}(1+1/\lambda_0) + \sqrt{\frac{1}{1-\lambda_0}}(1-1/\lambda_0)\right)^2$$

$$T_0 = \frac{GMm}{4r_0}\left[\left(\sqrt{\frac{1}{1+\lambda_0}} + \sqrt{\frac{1}{1-\lambda_0}}\right)^2 + \lambda_0^2\left(\sqrt{\frac{1}{1+\lambda_0}}(1+1/\lambda_0) + \sqrt{\frac{1}{1-\lambda_0}}(1-1/\lambda_0)\right)^2\right]$$

$$T_0 = \frac{GMm}{4r_0}\left[\left(\sqrt{\frac{1}{1+\lambda_0}} + \sqrt{\frac{1}{1-\lambda_0}}\right)^2 + \left(\sqrt{1+\lambda_0} - \sqrt{1-\lambda_0}\right)^2\right]$$

$$T_0 = \frac{GMm}{4r_0}\left[\left(\frac{1}{1+\lambda_0} + \frac{1}{1-\lambda_0} + 2\sqrt{\frac{1}{1+\lambda_0}}\sqrt{\frac{1}{1-\lambda_0}}\right) + \left(1+\lambda_0+1-\lambda_0 - 2\sqrt{1+\lambda_0}\sqrt{1-\lambda_0}\right)\right]$$

$$T_0 = \frac{GMm}{2r_0}\left[\frac{1}{1-\lambda_0^2} + \sqrt{\frac{1}{1-\lambda_0^2}} + 1 - \sqrt{1-\lambda_0^2}\right]$$

The potential energy U is given by

$$U = -GMm\left(\frac{1}{r_0+L} + \frac{1}{r_0-L}\right) = \frac{-GmM}{r_0}\left(\frac{1}{1+\lambda_0} + \frac{1}{1-\lambda_0}\right) = \frac{-GmM}{r_0}\left(\frac{2}{1-\lambda_0^2}\right)$$

Thus $H_0 = T_0 + U_0$ is given by

$$H_0 = \frac{GMm}{2r_0}\left[\frac{1}{1-\lambda_0^2} + \sqrt{\frac{1}{1-\lambda_0^2}} + 1 - \sqrt{1-\lambda_0^2} - \frac{4}{1-\lambda_0^2}\right] = \frac{GMm}{2r_0}\left[1 + \sqrt{\frac{1}{1-\lambda_0^2}} - \sqrt{1-\lambda_0^2} - \frac{3}{1-\lambda_0^2}\right]$$

In order to

$$-B = \sqrt{GMm^2 r_0}\left(\sqrt{1+\lambda_0} + \sqrt{1-\lambda_0}\right)$$

At the $\theta = 0$ equilibrium,

$$B = \bar{p}_\theta(\bar{\lambda}^{-2} - 1)$$

where

$$\bar{p}_\theta = \pm 2\ \bar{\lambda}^2\sqrt{\ GMm^2\bar{r}}\frac{\sqrt{1+\bar{\lambda}^2}}{\left(1-\bar{\lambda}^2\right)}$$

yielding

$$B = \pm 2\sqrt{\ GMm^2\bar{r}}\sqrt{1+\bar{\lambda}^2}$$

This implicitly determines $\bar{r}$ and $\bar{\lambda}$ in terms of r_0 and λ_0.

At the $\theta = \pi/2$ fixed point, again we have

$$B = \bar{p}_\theta(\bar{\lambda}^{-2} - 1)$$

but now

$$\hat{p}_\theta{}^2 = GMm^2\hat{r}\frac{4\hat{\lambda}^4}{\left(1+\hat{\lambda}^2\right)^{3/2}}$$

$$\hat{p}_\theta = \pm\sqrt{GMm^2\hat{r}}\frac{2\hat{\lambda}^2}{\left(1+\hat{\lambda}^2\right)^{3/4}}$$

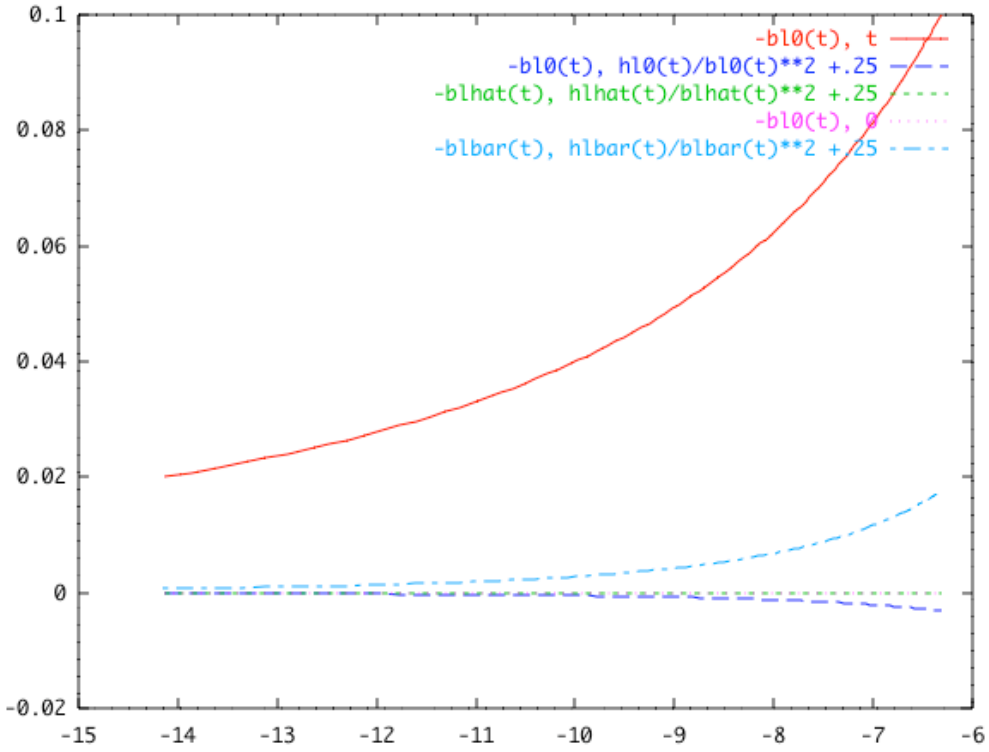
so that

$$B = \pm 2\sqrt{GMm^2\hat{r}}\frac{1-\bar{\lambda}^2}{\left(1+\hat{\lambda}^2\right)^{3/4}}$$

Now for any given r_0 and λ_0 we may determine B and thence $\bar{r}$, $\bar{\lambda}$, $\bar{H}$, $\hat{r}$, $\hat{\lambda}$ and $\hat{H}$.

numerical answers Since we have implicit functions, we have to use some numerical method to get a definite answer. Since B must have the same value for all cases, I used B as the independent variable. It was a bit more convenient to use $B_1 \equiv B/\sqrt{\lambda_0}$. The plot below shows λ_0 vs B_1 along with various H's. Instead of plotting H_0 itself, I plotted $H_{01} \equiv H_0/B_1^2 - .25$. This function was defined to make

the differences between H_0, $\hat{H}$ and $\bar{H}$ visible.



Plot of the function H_{01} defined above and the analogous functions $\hat{H}_1$ and $\bar{H}_1$. versus the dummy variable $-B_1$. The top curve is λ_0. The other three curves from top to bottom are $\bar{H}_1$, $\hat{H}_1$ and H_{01}, using the functions derived in the text.

The $\bar{H}_1$ should be the smallest of the three H curves, followed by H_{01} and $\hat{H}_1$. This is not the case; the $\bar{H}_1$ curve which should be the lowest one is instead the largest. This could easily be due to an algebra mistake above. If the H_{01} and $\hat{H}_1$ curves are correct, we would infer that the initial condition gives insufficient energy for escape, and the motion would be oscillatory. Venus has a $\lambda_0 \simeq 0.2$, though the plot only covers the region up to $\lambda_0 = 0.1$. The ordering of the energies at $\lambda_0 \simeq .2$ is the same as for $\lambda = .1$.