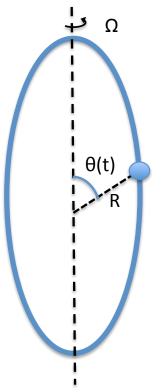
Answer these questions in the blue books provided, briefly explaining your reasoning. Do not consult books, notes, or electronic devices. Please circle your answers.

F.1 (12 points) Short answer

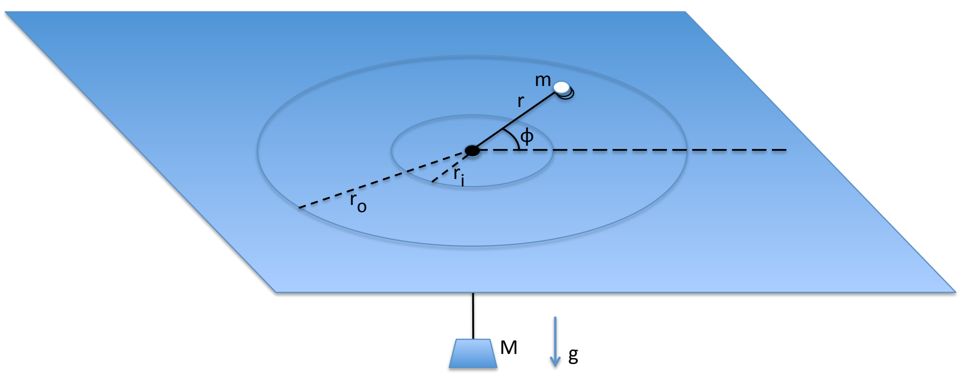


A particle of mass m moves freely on a vertical circular hoop that has radius R and is rotating at a fixed angular speed Ω about a vertical axis. We describe the motion in terms of the generalized co-ordinate $\theta(t)$ shown. There is no gravity.

a) Is the kinetic energy T equal to the Hamiltonian $\mathcal{H}$?
b) Is T conserved?
c) Is $\mathcal{H}$ conserved?

d) A certain force field $\vec{F}(x,y)$ can be written in Cartesian components as $f(x,y)\hat{x} + g(x,y)\hat{y}$. This force is conservative. In addition $\partial f/\partial y = 0$ at $x = y = 0$. What can you say about g at $x = y = 0$?

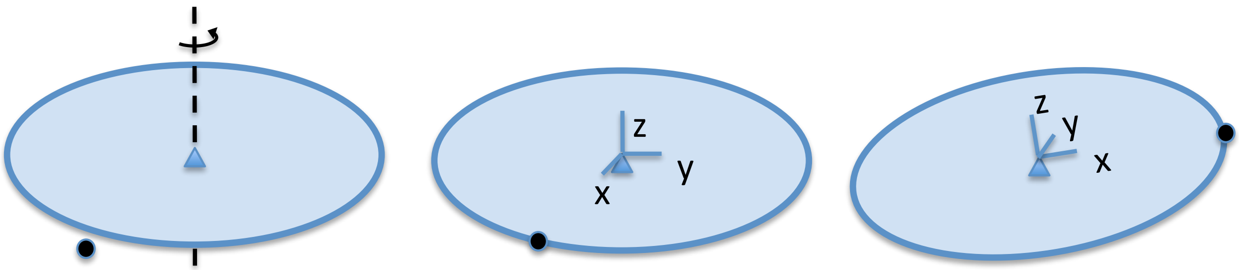
F.2 (15 points) Table gyrations



A mass m is free to slide without friction on a horizontal table. It is attached to a cord that runs through a hole in the table to a hanging mass M that is free to move vertically. The mass m is swinging around the hole and is also moving inward and outward. We denote the angular co-ordinate as ϕ. We observe that the mass moves radially between an inner distance r_i and an outer distance r_o.

a) What is the Lagrangian function $\mathcal{L}$ for this system, expressed using the generalized co-ordinates r and ϕ and their time derivatives $\dot{r}$ and $\dot{\phi}$?
b) Denoting the mechanical energy as E, Find $E/(Mgr_i)$ in terms of the ratio $b \equiv r_o/r_i$.
c) If the ratio b approaches 1, what is the limiting value of $E/(Mgr_i)$?

F.3 (20 points) Perturbed ring



A rigid horizontal circular ring of mass M and radius R is spinning about its vertical axis of symmetry at angular speed ω_0. It is spinning on a fixed pivot at its center (left picture). Then a small pellet of mass $m \ll M$ collides with the ring and sticks there. The pellet's motion before the collision was negligibly slow.

a) Calculate the inertia tensor **I** about the pivot point in co-ordinates where z is the axis of symmetry of the original ring and x is the direction towards the pellet.

b) What is the magnitude and direction of $\vec{\omega}$ relative to $\vec{\omega}_0$ immediately after the collision (so that the ring has not had time to rotate appreciably, middle picture)?

c) The kinetic energy T changes from its initial value T_0 due to the collision. What is T/T_0? Did the energy increase or decrease? Express your answer in terms of the mass ratio m/M.

A short time later (right picture) the disk has rotated so that $\vec{\omega}$ lies in the y-z plane.

d) At this moment, what is the magnitude of the angular momentum $|L|$ relative to its original value $|L_0|$ before the pellet was attached? Express your answer in terms of the mass ratio m/M.

e) At this same moment, what is $|\omega_y/\omega_z|$? Express your answer in terms of the mass ratio m/M.

F.4 (10 points) An orbital quantity

The Hamiltonian $\mathcal{H}$ for motion of a planet of mass m a distance r from a star exerting a gravitational force $-\gamma/r^2$ is given by

$$\mathcal{H} = \frac{1}{2m}\, p_r^2 + \frac{1}{2mr^2}\, {p_\phi}^2 - \gamma/r$$

Here p_r is the generalized momentum conjugate to the radial co-ordinate r and p_ϕ is the momentum conjugate to the angular co-ordinate ϕ. A quantity of interest in orbital mechanics is $B \equiv p_r\ p_\phi$. Hamilton's equations allow us to express the time derivatives $\dot{r}, \dot{\phi}, \dot{p}_r, \dot{p}_\phi$ in terms of r, ϕ, p_r, p_ϕ themselves.

a) What is the time derivative of p_ϕ in terms of these co-ordinates and momenta?

b) Express $\dot{B}$ in terms of constants of the motion and r.

Solution:

F.1

a) Kinetic energy T is $\frac{1}{2}m(R\sin\theta\,\Omega)^2 + \frac{1}{2}mR^2\dot{\theta}^2$. Hamiltonian $\mathcal{H}$ is $\dot{\phi}\partial T/\partial\dot{\phi} - T$. Because of the Ω^2 part of T $\mathcal{H} \neq T$.

b) No. the thing forcing the constant Ω rotation does work on the system and changes its mechanical energy.

c) Yes. $d\mathcal{H}/dt = \partial\mathcal{L}/\partial t = \partial T/\partial t = 0$.

d) If $\vec{F}$ is to be conservative, its curl must vanish everywhere. $0 = \nabla \times F = \hat{z}(\partial f/\partial y - \partial g/\partial x)$. Since the first term vanishes, the second must also. So we know $\partial g/\partial x = 0$.

F.2

a) $\mathcal{L} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - Mgr$ (We note that we are free to take any height to be 0 in the gravitational potential. Here we take it to be the height of the hanging mass when $r = 0$.)

b)

$$E = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + Mgr$$

At the r_i or r_o the $\dot{r}$ term vanishes. The $\dot{\phi}$ term can be expressed in terms of the conserved angular momentum L: $L = mr^2\dot{\phi}$. thus

$$E = \frac{L^2}{2m{r_i}^2} + Mgr_i = \frac{L^2}{2m{r_o}^2} + Mgr_o$$

Using both equations we may eliminate L to find E in terms of r_i and r_o: $\frac{L^2}{2m{r_i}^2} = E - Mgr_i$, so that

$$L^2 = 2m({r_i}^2E - Mg{r_i}^3) = 2m({r_o}^2E - Mg{r_o}^3)$$

Thus

$${r_i}^2E - Mg{r_i}^3 = {r_o}^2E - Mg{r_o}^3$$

or

$$E(r_i^2 - r_0^2) = Mg(r_i^3 - r_o^3)$$

or using $b = r_0/r_i$

$$E = Mgr_i\frac{b^3-1}{b^2-1}$$

c) Letting $b = 1+\epsilon$, $b^3 \to 1+3\epsilon$, $b^2 \to 1+2\epsilon$, so that $E \to \frac{3}{2}Mgr_i$.

F.3

a)

i) For I_{zz} all the ring mass is at distance R from the axis. The pellet contributes an additional mR^2 so $I_{zz} = (M+m)R^2$.

ii) For I_{xx} there is no contribution from the pellet.

$$I_{xx} = \frac{M}{2\pi R}\int x^2 = \frac{1}{2}\frac{M}{2\pi R}\int (x^2+y^2) = \frac{1}{2}I_{zz} = \frac{1}{2}MR^2.$$

iii) For I_{yy} the pellet contributes mR^2, so $I_{yy} = \frac{1}{2}MR^2 + mR^2$

All the off-diagonal I's are 0.

b) z principal axis is still pointing in the direction of L; $L = L_0 = I_{zz0}\ \omega_0 = MR^2\omega_0$. But $L = L_z = I_{zz}\omega$. So $MR^2\omega_0 = MR^2(1+m/M)\omega$. and $\omega = \omega_0/(1+m/M)$. Since I is diagonal, ω_x would be L_x/I_{xx}. But $L_x = 0$, so $\omega_x = 0$; likewise $\omega_y = 0$: $\vec{\omega}$ points along z.

c) $\vec{\omega}$ is in the z direction, so

$$T = \frac{1}{2}I_{zz}\omega^2 = \frac{1}{2}MR^2(1+m/M)(\frac{\omega_0}{1+m/M})^2 = \frac{1}{2}\frac{\omega_0^2}{1+m/M} = \frac{T_0}{1+m/M}.$$

T decreases.

d) $\vec{L}$ is conserved; thus $|L| = |L_0|$.

e) At this position using the x, y, z principal axes

$$T = L_z^2/(2I_{zz}) + L_y^2/(2I_{yy}).$$

On the other hand, $T = L_0^2/(2I_{zz})$. Simplifying and using $|L| = L_0$

$$\begin{aligned} 1 &= (L_z/|L|)^2 + (L_y/|L|)^2(I_{zz}/I_{yy}) = (L_z/|L|)^2 + (L_y/|L|)^2 + (L_y/|L|)^2(I_{zz}/I_{yy}-1) \\ &= 1 + (L_y/|L|)^2(I_{zz}/I_{yy}-1) \end{aligned}$$

Since $I_{zz} > I_{yy}$ the only solution is $L_y = 0$; thus ω_y/ω_z is also zero.

The disk in fact *does not precess* when it is perturbed in this way.

F.4

a) $\dot{p}_\phi = \partial\mathcal{H}/\partial\phi$. This partial derivative is to be taken in phase space with $r, p_\phi and p_r$ fixed. Since ϕ does not appear in $\mathcal{H}$ above, the derivative is zero and $\dot{p}_\phi = 0$.

b) One may compute $\dot{B}$ from the Poisson Bracket formula, which amounts to the chain rule.

$$\dot{B} = \partial B/\partial p_\phi\ \dot{p}_r + \partial B/\partial p_\phi\ \dot{p}_\phi$$

The second term vanishes, since p_ϕ is constant. The $\dot{p}_r = -\partial\mathcal{H}/\partial r = -{p_\phi}^2/(mr^3) - \gamma/r^2$ $\partial B/\partial p_r = p_\phi$. Combining,

$$\dot{B} = p_\phi\left({p_\phi}^2/(mr^3) - \gamma/r^2\right)$$

Since p_ϕ is a constant of the motion, this expression has the requested form.