January 3, 2012

Physics 185: Witten's January 1 Partial derivative query

The query

I have a function $f(x, y)$, where x and y are two cartesian co-ordinates. I define the polar co-ordinate r in the usual way: $r = \sqrt{x^2 + y^2}$. Is there any difference between the partial derivative of f with respect to x at constant y versus the partial derivative of f with respect to x at constant r? Would you be able to find the relation between these two partial derivatives?

The answer

Partial derivatives like $\partial f / \partial x$ depend on what is being held constant. We often indicate what is to be held constant by a subscript, *eg.*

$$\left(\frac{\partial f}{\partial x}\right)_y$$

Given a function $f(x, y)$ of Cartesian co-ordinates x, and y, we are asked to compare*

$$\left(\frac{\partial f}{\partial x}\right)_y \quad \text{with} \quad \left(\frac{\partial f}{\partial x}\right)_r, \tag{1}$$

where $r \equiv \sqrt{x^2 + y^2}$. A simple way to do this is to recognize that the total change in f, df under a small change of variables can be written

$$df = \left(\frac{\partial f}{\partial x}\right)_y \, dx + \left(\frac{\partial f}{\partial y}\right)_x \, dy \tag{2}$$

or

$$df = \left(\frac{\partial f}{\partial x}\right)_r \, dx + \left(\frac{\partial f}{\partial r}\right)_x \, dr \tag{3}$$

We may compare $\left(\frac{\partial f}{\partial x}\right)_y$ with $\left(\frac{\partial f}{\partial x}\right)_r$ by expressing dy in terms of dx and dr.

$$2x \; dx + 2y \; dy = 2r \; dr$$

so that

$$dy = r/y \; dr - x/y \; dx$$

Using this expression for dy, our first equation for df may be written

$$df = \left(\frac{\partial f}{\partial x}\right)_y \, dx + \left(\frac{\partial f}{\partial y}\right)_x \, [r/y \; dr - x/y \; dx]$$

Or

$$df = \left[\left(\frac{\partial f}{\partial x}\right)_y - \left(\frac{\partial f}{\partial y}\right)_x x/y\right] dx + \left(\frac{\partial f}{\partial y}\right)_x r/y \; dr \tag{4}$$

Comparing Equation 3 with Equation 4, we infer that

$$\left(\frac{\partial f}{\partial x}\right)_r = \left[\left(\frac{\partial f}{\partial x}\right)_y - \left(\frac{\partial f}{\partial y}\right)_x x/y\right]$$