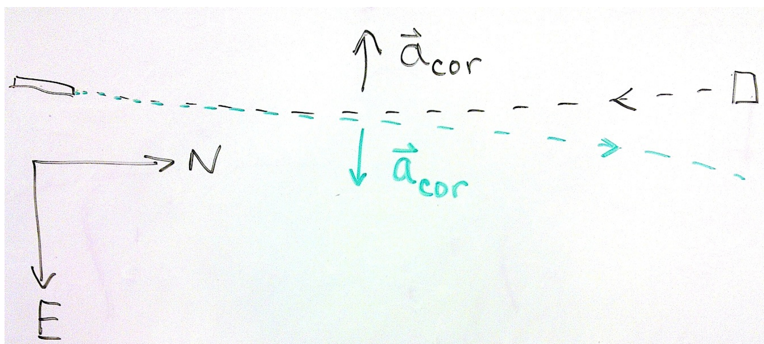
Several things were garbled in this lecture. These notes are meant to correct and clarify.

the laser puzzle.

I discussed the puzzle posed in Friday's lecture. A laser on the roof of Kersten is aimed at a distant object (a mast on the Sears Tower) via a telescopic sight. We suppose the mast is directly north of the laser. Then the laser is fired toward the object. The question is whether the beam hits the object or whether inertial forces from the earth's motion interfere with the aim. I ignored gravity and the centrifugal force and considered only the Coriolis acceleration $\vec{a}_{\text{cor}} = \dot{\vec{r}} \times \vec{\Omega}$. We consider the Coriolis acceleration of the light rays coming into the telescope from the mast.



The direction of $\dot{\vec{r}}$ is southward. The direction of $\vec{\Omega}$ is in the North-Up plane (as found in Wednesday's lecture). Thus $\vec{a}_{\text{cor}}$ is perpendicular to this plane, the East-West direction. Using the right hand rule, $\vec{a}_{\text{cor}}$ points West.

So the south-moving ray experiences a westward Coriolis acceleration. In order to reach the telescope, directly to the south, the ray must start out directed slightly eastward. On entering the telescope, it is moving slightly westward. Now we consider the laser beam, we send it in the direction towards the light received by the telescope. Thus the beam is traveling north and slightly East. It too experiences a Coriolis acceleration $\dot{\vec{r}} \times \vec{\Omega}$. Since $\dot{\vec{r}}$ is now reversed, $\vec{a}_{\text{cor}}$ is also reversed (green ray in figure): the acceleration is almost Eastward, (since the outgoing ray is almost northward). The initial beam is also directed slightly eastward; the $\vec{a}_{\text{cor}}$ acceleration bends it even further eastward.

One may find the magnitude of the deflection by including only the dominant East-West part of $\vec{a}_{\text{cor}}$. If the beam travels 10 km, the deflection works out to be of the order of 10-20 microns.

The gravitational and centrifugal accelerations are simpler to treat. These accelerations are independent of $\dot{\vec{r}}$. Thus return ray is deflected with the same acceleration as the incoming ray. The return ray's motion is just the time-reversal of the incoming ray's motion. So these two forces produce no aiming error—unlike the Coriolis force.

Angular momentum of a rigid body

Any rigid body may be viewed as an assembly of small masses m with positions $\vec{r}_\alpha$, such that all distances between two such masses α and β are constant in time. We noted that the most general possible motion of such a body is a time-dependent translation and a time-dependent rotation with some angular velocity $\vec{\omega}(t)$.

Since we are describing the motion in terms of rotational co-ordinates, we naturally need to express the corresponding momenta in terms of these co-ordinates. As we know, these momenta are simply the angular momentum vector $\vec{L}$. As noted in lecture and text, this $\vec{L}$ can be decomposed simply in terms of the center-of-mass angular momentum $M\ \vec{R}_{cm} \times \vec{P}$ and the angular momentum $\vec{L}_{\mathcal{S}}$ computed in a co-ordinate system $\mathcal{S}$ where the the center of mass is at rest at the origin:

$$\vec{L} = M\ \vec{R}_{cm} \times \vec{P} \quad + \quad \vec{L}_{\mathcal{S}}$$

To see how $\vec{L}$ is related to $\vec{\omega}$ we may thus suppose that the center of mass is at rest at the origin, so $\vec{L}$ is just $\vec{L}_{\mathcal{S}}$. Then the velocity of mass α is simply $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha$. The angular momentum $\vec{L}$ is then given by

$$\vec{L} = \sum_\alpha\ m\ \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = m\ \sum_\alpha \vec{r} \times (\vec{\omega} \times \vec{r}_\alpha)$$

Evidently $\vec{L}$ is proportional to $\vec{\omega}$, but in a complicated-looking way. The text computes it using x-y-z notation. The notation below is a bit more compact. We first reduce the triple product using the BAC–CAB formula:

$$\vec{L} = m\ \sum_\alpha \vec{\omega}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{\omega})$$

We can simplify this proportionality by computing $\vec{L}$ by components. We label the components by 1, 2, 3 and then consider the component $\vec{L}_k$. We also express the vectors on the right side in terms of components:

$$\vec{L}_k = m\ \sum_\alpha \vec{\omega}_k(\vec{r} \cdot \vec{r}) - \vec{r}_k(\vec{r} \cdot \vec{\omega}) = m\ \sum_\alpha \sum_\ell \vec{\omega}_k\ \vec{r} \cdot \vec{r} - \vec{r}_k\ \vec{r}_\ell\ \vec{\omega}_\ell$$

It is useful to introduce the so-called Kroenecker delta notation δ_{ij}. This δ_{ij} defined to be 1 when $i = j$ and 0 otherwise. Then we can represent $\vec{\omega}_k$ as $\sum_\ell \vec{\omega}_\ell \delta_{k\ell}$. Thus,

$$\vec{L}_k = m\ \sum_\ell \left[\sum_\alpha \delta_{k\ell}\ \vec{r} \cdot \vec{r} - \vec{r}_k\ \vec{r}_\ell \right]\ \vec{\omega}_\ell$$

We now see that the proportionality constant in [...] involves both the component index of $\vec{\omega}$ and that of $\vec{L}$. We label the part of [...] involving components k and ℓ as $\mathbf{I}_{k\ell}$. That is,

$$\mathbf{I}_{k\ell} = m\ \sum_\alpha \delta_{k\ell}\ \vec{r} \cdot \vec{r} - \vec{r}_k\ \vec{r}_\ell \tag{1}$$

Then the relation between $\vec{L}$ and $\vec{\omega}$ simplifies to

$$\vec{L}_k = \sum_\ell\ \mathbf{I}_{k\ell}\ \vec{\omega}_\ell$$

This formula has the form of a matrix $\mathbf{I}$ with components $\mathbf{I}_{k\ell}$ multiplying the vector $\vec{\omega}$.

$$\vec{L} = \mathbf{I}\ \vec{\omega}$$

The entire multicomponent object $\mathbf{I}$ is called the inertia tensor of the object with the given origin.

We now examine the matrix $\mathbf{I}$ in more detail. It has "diagonal" components $\mathbf{I}_{11}, \mathbf{I}_{22}, \mathbf{I}_{33}$. Both the first and second term in Eq. (1) contribute to these diagonal components. The first term is ${r_1}^2 + {r_2}^2 + {r_3}^2$. The second term subtracts out the indexed component, leaving only the other two. Thus $\mathbf{I}_{11} = {r_2}^2 + {r_3}^2$, and so forth. These diagonal components are known as the three "moments of inertia". The non-diagonal components of $\mathbf{I}$, whose indices are different, are called the products of inertia. Only the second term in Eq (1) contributes to these.

The sum over masses α of a continuous body is naturally expressible as an integral. In general such integrals are complicated to perform. However, one can often simplify them greatly when the body has spatial symmetries. One simplifying feature is valid for any body: The matrix $\mathbf{I}$ is *symmetric*: $\mathbf{I}_{k\ell} = \mathbf{I}_{\ell k}$. One verifies this by noting that Eq. (1) has the same value when ℓ and k are interchanged. The next symmetry to note is that the off-diagonal components are *odd* functions of the two r components involved. This symmetry allows one to show that a given $\mathbf{I}_{k\ell} = 0$ for certain objects. For definiteness we consider the component $k = 1$. Now we consider a body that is *mirror-symmetric* in the 1 direction. That is, for every mass in the object at point (r_1, r_2, r_3), there is an identical mass at $(-r_1, r_2, r_3)$. The contribution of these two masses to, say, $\mathbf{I}_{12}$, is $m(r_1 r_2 + (-r_1) r_2)$; it sums to zero. We may sum over all the masses of the symmetric object pairwise in this way, so that the total $\mathbf{I}_{12}$ must be zero. The same reasoning applies to $\mathbf{I}_{13}$. We conclude that any object with a mirror symmetry in the plane at the origin perpendicular to the k axis has $\mathbf{I}_{k\ell} = 0$ for all off-diagonal elements involving k.