Physics 185 Midterm in class 1 Feb 2012.   name:..

Answer these questions in the blue books provided, briefly explaining your reasoning. Do not consult books, notes, or electronic devices. Please circle your answers.

M.1 (10 points) short answer

a) a bead of mass m is at the top of a frictionless, stationary vertical circular hoop of radius R in earth's gravity. It is displaced slightly and begins to slide along the hoop. What is its speed at the bottom?

b) A toy rocket expels mass at a rate $\dot{m}$ with a speed v_{ex} (relative to the rocket). The rocket pushes horizontally against your stationary palm and exerts a force F on you. If you allow the rocket to move forward at speed v, how does the force F change?

M.2 (16 points) bent rod

Bending a rod adds potential energy to it. The energy in a short segment of length Δs at position s along the rod is $\Delta U = \frac{B}{2}(d\theta/ds)^2\ \Delta s$, where θ is the angle of the rod at s and B is a constant. A certain rod of length L is mounted horizontally to a wall by one end. A weight exerts a downward force W at the other end. Take θ to be angle relative to the horizontal.

a) Given a bending profile $\theta(s)$ express the potential energy of the system as an integral expressed in terms of $\theta(s)$ and its derivatives.

b) Find a differential equation for the shape of the rod.

c) Find the function $\theta(s)$ assuming that the weight is small so that θ remains very close to horizontal. Use the fact that $d\theta/ds$ vanishes at the free end of the rod.

M.3 (14 points) trapped on a sphere

A particle of mass m is constrained to move on a sphere of radius R. Its angle with respect to the vertical is θ. Its angle with respect to the x axis in the horizontal plane is ϕ. In terms of these co-ordinates the speed of the particle v can be written

$$v^2 = R^2\ \dot{\theta}^2 + R^2 \sin^2\theta\ \dot{\phi}^2$$

This particle feels an external potential $U = \frac{1}{2}k\cos^2\theta$,

a) At some moment the particle passes through the equatorial plane where $\theta = \pi/2$. Find equations for $\ddot{\theta}$ and $\ddot{\phi}$ at this point.

b) Find a function of $\phi, \theta, \dot{\theta}$, and/or $\dot{\phi}$ that is conserved, so that it remains constant during the motion.

Solution

M.1

a) The change of kinetic energy is minus the change of potential energy ie, $2mgR = \frac{1}{2}mv^2$, so $v = 2\sqrt{gR}$.

b) The force $F = \dot{m}v_{ex}$ independent of the rocket's motion. Since the hand moves at fixed speed v and is not accelerating, the force on the hand is also $\dot{m}v_{ex}$. It is unchanged by moving the hand.

M.2

a) Since the rod bends down from its anchoring point at $y = 0$, we take both y and θ to be negative. $U = \int_0^L ds(\frac{1}{2}B\theta'^2) + Wy_{\text{end}}$ We must still specify y_{end} in terms of $\theta(s)$: $y_{\text{end}} = \int_0^L ds\ dy/ds = \int \sin\theta ds$. so

$$U = \int_0^L ds\left(\frac{1}{2}B\theta'^2 + W\sin\theta(s)\right)$$

b) Denoting the integrand as $f(\theta, \theta')$, the Euler Lagrange equations give the condition for a minimal U. $\partial f/\partial\theta = W\cos\theta$; $\partial f/\partial\theta' = B\theta'$, and the E-L equation says

$$B\theta'' - W\cos\theta = 0$$

c) If $\theta \ll 1$, then $\cos\theta \to 1$ and $\theta'' = W/B$. Then $\theta(s) = As + \frac{1}{2}(W/B)s^2$. where A is some constant for which U is minimal

why θ' vanishes at the free end

$$0 = dU/dA = \frac{d}{dA}\left(\int ds\frac{1}{2}B(\theta')^2 + W\theta(s)\right)$$

$$= \frac{d}{dA}\left(\int ds\frac{1}{2}B(A + W/B\ s)^2 + W(As + W/B\ s^2)\right)$$

$$0 = \int B(A + W/Bs) + W\ s$$

$$0 = \int dsBA + 2Ws$$

Evidently the integrand must be zero at the midpoint (or you can just do the integrals to find) $BA + 2W(L/2) = 0$ so $A = -WL/B$ so θ' vanishes at $s = L$, as stated in the problem.

Thus finally

$$\theta(s) = -WL/Bs + \frac{1}{2}W/B\ s^2 = (W/B)s(\frac{1}{2}s - L)$$

M.3

a)

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$\mathcal{L} = T - U = \frac{1}{2}mv^2 - \frac{1}{2}k\cos^2\theta = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \frac{1}{2}k\cos^2\theta,$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \partial T/\partial\dot{\theta} = mR^2\dot{\theta}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = mR^2\sin\theta\cos\theta\dot{\phi}^2 - k\cos\theta\sin\theta = \sin\theta\cos\theta(mR^2\dot{\phi}^2 - k)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mR^2\sin^2\theta\dot{\phi}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Equations of motion can thus be written, for θ,

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta} = \left(\sin\theta\cos\theta(mR^2\dot{\phi}^2 - k)\right)$$

so at the point $\theta = \pi/2$, $\sin\theta = 1, \cos\theta = 0$ and

$$\ddot{\theta} = 0$$

For ϕ,

$$0 = \frac{d}{dt}mR^2\sin^2\theta\dot{\phi} = 2mR^2\sin\theta\cos\theta\dot{\phi} + m\sin^2\theta\ddot{\phi}$$

At $\theta = \pi/2$, we have $\boxed{\ddot{\phi} = 0}$.

b) Evidently the ϕ equation of motion implies that $\sin^2\theta\dot{\phi}$ is a constant in time, a constant of the motion.