This handout reproduces and clarifies my lecture of March 2 about Poisson brackets.

Chapter 13 and previous lectures show that Hamilton's equations give the time derivatives for the phase space co-ordinates $q_1...q_n, p_1,,,p_n$ as a derivative of the Hamiltonian function. In other words the time derivatives were expressed as a kind of generalized gradient. It is analogous to the way conservative forces are expressed as a gradient of a potential: $F = -\nabla U$. This gradient form eventually allowed us to formulate the motion in terms of Hamilton's extremal principle. This eventually allowed us to express the laws of motion for arbitrary co-ordinates q_i. Now by using phase space co-ordinates, we are able to express the whole equations of motion in the form of a gradient-like operation. As with the conservative forces, this gradient-like form allows gives us additional flexibility in our choice of variables to describe the system. This lecture describes how this flexibility comes about. The first step is to show how this gradient-like operation restricts the kind of motion that can occur in phase space, independent of the Hamiltonian we are using. In the lecture we represented the whole list of q's as $\{q\}$ and the whole list of p's as $\{p\}$. My $\{q\}$ and $\{p\}$ are represented in the book as **q** and **p**.

Using Hamilton's equations

$$\dot{q}_i = \partial\mathcal{H}/\partial p_i \quad \dot{p}_i = -\partial\mathcal{H}/\partial q_i$$

we can readily construct an expression that gives the derivative of an arbitrary physical quantity $f(\{q\},\{p\})$. Using the chain rule

$$\dot{f} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} \tag{1}$$

This is true for any $\mathcal{H}$ we might imagine. This expression succinctly states how any physical quantity changes under the action of another physical quantity, viewed as the Hamiltonian. This binary operation can clearly be computed for any two physical quantities. It is known as the *Poisson bracket*. The Poisson bracket of the quantities f and g is denoted $[f, g]$.

The Poisson bracket gives a convenient way to describe the new flexibility of defining co-ordinates and momenta that are possible in phase space. Before proceeding we note some of its mathematical properties.

- $[f, g]$ is *antisymmetric*: $[g, f] = -[f, g]$.
- $[f, g]$ is *bilinear*: for any linear combination of two physical quantities f_1 and f_2,

  $$[a_1 f_1 + a_2 f_2, g] = a_1[f_1, g] + a_2[f_2, g],$$

  and similarly for g.
- *Product rule*: If f is a product of two physical quantities u and v, then $[uv, g] = u[v, g] + v[u, g]$. Similarly for g.
- If f and g are simply q_i or p_i, their Poisson brackets take a simple form that follows from the definition of the partial derivatives.

  $$[q_i, q_j] = 0; \quad [p_i, p_j] = 0; \quad [q_i, p_j] = \delta_{ij} \tag{2}$$

This δ_{ij} is the function we introduced earlier: 1 if $i = j$ and 0 otherwise. I ask you to verify some of these properties in Problem set 9

Transformations generated by $[f, g]$

We can think of $[f, g]$ as defining a transformation of the function f, governed by g. If g is the Hamiltonian $\mathcal{H}$, then $[f, \mathcal{H}]$ generates the new function that describes f after a short time Δt:

$$f(\{q\}, \{p\}, t + \Delta t) = f(\{q\}, \{p\}, t) + \Delta t\ [f, \mathcal{H}] \tag{3}$$

For example, we can consider a harmonic oscillator with mass m and spring constant k. Then using x for its position and p for its momentum, the Hamiltonian $\mathcal{H} = \frac{1}{2m}p^2 + \frac{k}{2}x^2$. Now we take f to be the position x of an oscillator which started at position x_0 and momentum p_0, then the transformed $x(x_0, p_0, t + \Delta t) = x(x_0, p_0, t) + \Delta t[x, \mathcal{H}]$. Evidently this is an infinitessimal transformation. In order to change f significantly, we would have to repeat the transformation many times, each time using the f from the previous time. By such repetition we can ultimately obtain f as a function of the initial state of the system $\{q\}(0)$, $\{p\}(0)$. In other words we can find any physical quantity at time t given the initial state of the system.

Clearly we can make an analogous transformation replacing $\mathcal{H}$ by any physical quantity g. We will make several such transformations below, so we'll define a notation for the transformed quantity:

$$\tilde{f}_{\epsilon,g} \equiv f + \epsilon\ [f, g] \tag{4}$$

To show that these transformations are meaningful, we consider what happens when we use one of the $\{p\}$, *eg.* p_1, in place of $\mathcal{H}$. We see immediately from the definition of $[f, g]$ that $[f, p_1] = \partial f/\partial q_1$; all the other terms vanish.

$$\tilde{f}_{\epsilon,p_1} = f + \epsilon\ [f, p_1] = f + \epsilon\ \frac{\partial f}{\partial q_1} \tag{5}$$

But this is this is just the formula for an infinitesimal shift in the q_1 variable: *i.e.,*

$$\tilde{f}_{\epsilon,p_1}(\{q\}, \{p\}) = f(q_1 + \epsilon, q_2, ..., q_n, \{p\}),$$

as one can verify by just Taylor-expanding the right hand side. Thus the transformation generated by p_i corresponds to an infinitesimal advance of the corresponding q_i. In Problem set 9, you see how this works for the polar angle variable ϕ which describes gives the rotational orientation of the system.

As with the Hamiltonian transformation of Eq. (3), one can create a finite transformation for any f and g by repeating the infinitesimal transformation many times. Any such transformation needs a variable analogous to t, telling how much transforming is to be done. For example, if the generating quantity is the momentum p_ϕ conjugate to the rotational angle ϕ then the finite transformation generated by p_ϕ, specifies by what angle the system has been rotated.

We may choose any physical quantity to play the role of the transformed variable f, including the Hamiltonian. Then Eq. (5) says

$$[\mathcal{H}, p_1] = \partial\mathcal{H}/\partial q_1.$$

Sometimes we are able to find variables $\{q\}$ such that $\mathcal{H}$ is independent of q_1 (such as the angle ϕ in a central force problem). The book calls such a q an ignorable variable. For such variables, the partial derivative must vanish, so that $[\mathcal{H}, p_1] = 0$. As noted above this means $[p_1, \mathcal{H}]$ is also zero. But his quantity is the time derivative of p_1: p_1 is a constant of the motion. We have seen this result repeatedly starting with the Lagrangian formalism. It emerges in a simple way in the Poisson bracket notation. Naturally it is useful even when the Poisson bracket in question has some known, nonzero value. In general this argument shows that the infinitesimal change of f generated by g is the same (up to a sign) as the change of g generated by f.

Poisson brackets preserved under Poisson transformations

Just as we may compute the Poisson bracket of any f and g, so too we may compute the Poisson bracket of the *transformed* f and g. We now consider how these two Poisson brackets are related. We will use $h(\{q\},\{p\})$ to generate the transformation and then consider $[\tilde{f}_{\epsilon,h}, \tilde{g}_{\epsilon,h}]$. This object is straightforward to compute using Eq. (4) since [...] is linear:

$$[\tilde{f}_{\epsilon,h},\ \tilde{g}_{\epsilon,h}] = [f,g] + \epsilon\,(\ [f,[g,h]] + [[f,h],g]\ ) + \mathcal{O}(\epsilon^2) \tag{6}$$

As with all infinitesimal quantities the $\mathcal{O}(\epsilon^2)$ part is indefinitely small and may be neglected. The result we seek involves Poisson brackets of Poisson brackets. Happily one may work out a convenient relation between such quantities using the basic properties marked by • above. It is called the Jacobi identity and is true for any three physical quantities f, g, and h.

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

That is, the sum of $[f,[g,h]]$ and its cyclic permutations is zero. We note that the first term in (...) in Eq. (6) matches the first term above. The second term also matches if we note that $[[f,h],g] = -[[h,f],g] = +[g,[h,f]]$. Thus

$$[\tilde{f}_{\epsilon,h}, \tilde{g}_{\epsilon,h}] = [f,g] + \epsilon\,(-[h,[f,g]]) = [f,g] + \epsilon[[f,g],h] \tag{7}$$

For some f and g the result is simple. If f and g are any of the $\{q\}$ or $\{p\}$, then $[f,g]$ is either 0 or 1; all its derivatives are zero. Thus $[[f,g],h] = 0$ for any h. Then the the ϵ term simply vanishes: that is the $[f,g]$ remains *invariant* under a Poisson transformation for such f and g.

Now we consider what happens when we make further infinitessimal transformations. For this purpose I'll denote $[\tilde{f}_{\epsilon,h}, \tilde{g}_{\epsilon,h}]$ as simply $[f,g]_\epsilon$. We restrict our attention to f and g such that $[f,g]_\epsilon = [f,g]$. Now we perform the same transformation again to obtain $[f,g]_{2\epsilon}$.

$$[f,g]_{2\epsilon} = [f,g]_\epsilon + \epsilon\ [[[f,g]_\epsilon, h]$$

Noting that $[f,g]$ remained invariant on the first iteration, we find

$$[f,g]_{2\epsilon} = [f,g] + \epsilon\ [[f,g],h] = [f,g]$$

The Poisson bracket remains invariant. Clearly it will remain invariant under further iteration, so that it is true of finite transformations as well as infinitesimal ones. Furthermore, we need not keep h fixed as we repeat our transformations. Since $[[f,g],h] = 0$ under any h, given our restricted choice of f, and g. It clearly remains invariant under an arbitrary Poisson transformation.

This property shows that our initial $\{q\}$ and $\{p\}$ have a special property. That is independent of the Hamiltonian of our system. We may construct the $\{p\}$ for our chosen $\{q\}$ using some Lagrangian or other. But once we have these $\{p\}$, their Poisson brackets must satisfy Eq. (2). They must continue to do so under any Poisson transformation. We now denote the image of q_1 under such a transformation as $Q_1(\{q\},\{p\})$, and we define the other images similarly as $\{Q\}$ and $\{P\}$. As we have shown, the original Poisson bracket relations hold for the $\{Q\}$ and $\{P\}$. We denote any set of variables $\{Q\},\{P\}$ that satisfy Eq. (2) as *canonical.* We repeat that the property of being canonical does not depend on the system one is studying. For example for a particle in three-dimensional space in any kind of potential, $x, y, z, m\dot{x}, m\dot{y}, m\dot{z}$, are always a canonical set of variables. This property is built into the geometry of phase space. We have seen similar geometric properties in the more familiar context of vectors. Here the inner product of any two vectors is unchanged under any rotation of the space. Any rotated basis is equivalent to any other in describing dynamics. In the current context the role of the inner product is played by the Poisson bracket, although the Poisson bracket is much different than an inner product!

Canonical variables

For ordinary vectors we know that any rotated basis (not time dependent) is equivalent to any other, but do we know the analogous statement for canonical variables? That is, will we obtain the same dynamics using the $\{Q\},\{P\}$ as we would with the original $\{q\},\{p\}$? Since we can express dynamics using Poisson brackets, this amounts to asking whether we may compute these brackets using the $\{Q\}$ and $\{P\}$. We may certainly construct the analog of the Poisson bracket of physical quantities f and g using these $\{Q\}$ and $\{P\}$. We will dub this physical quantity as the "Fish bracket"* of f and g, and denote it by $[\![f,g]\!]$:

$$[\![f,g]\!] \equiv \sum_{i=1}^{N} \frac{\partial f}{\partial Q_i}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial Q_i} \tag{8}$$

In order for $\{Q\},\{P\}$ to be equivalent to $\{q\},\{p\}$, we need to show that $[\![f,g]\!] = [f,g]$. We first show it for one complete set of variables, namely the $\{Q\},\{P\}$ themselves. By definition the $\{Q\},\{P\}$ obey Eq.(2) using the original Poisson brackets:

$$[Q_i, Q_j] = 0; \quad [P_i, P_j] = 0; \quad [Q_i, P_j] = \delta_{ij} \tag{9}$$

But the same is true using the Fish brackets, just from their definition (8), *i.e.,* :

$$[\![Q_i, Q_j]\!] = 0; \quad [\![P_i, P_j]\!] = 0; \quad [\![Q_i, P_j]\!] = \delta_{ij}$$

It remains to show that $[\![f,g]\!] = [f,g]$ for arbitrary physical quantities f, and g. This is true, but the proof is usually reserved for more advanced courses. Here is a hint of the reasoning. Suppose that $\{q\},\{p\}$ describe the current state of the system. Any current state must arise from some previous state a time t earlier under some given Hamiltonian h. Each of the q_i evolved from some corresponding initial co-ordinate, that we shall call Q_i. Likewise for the each p_i. Because this system is described by Hamilton's equations, the problem of determining the final state that arose from some given initial state is equivalent to the reverse problem of tracing the initial state that would have arisen from a given final state. To solve the latter problem, one simply propagates Hamilton's equations backwards in time. That is, the $\{q\},\{p\}$ and $\{Q\},\{P\}$ must be an equivalent set of variables. In particular, Hamilton's equations must be true using the $\{Q\},\{P\}$ variables. From this is follows that any time derivative may be computed in the $\{Q\},\{P\}$ in the same way as with the $\{q\},\{p\}$. That means $\dot{f}$ can be calculated using $[\![f,h]\!]$ as well as $[f,h]$. This is true for any f and h. Thus the two types of bracket are equal.

This means that any set of variables generated by Poisson transformations from some canonical $\{q\},\{p\}$, is physically equivalent to these $\{q\},\{p\}$. Hamilton's equations are valid for the new variables, as for the old. We are free to choose a canonical co-ordinates as we wish, in order to simplify the problem at hand. This flexibility is the chief insight of the Hamiltonian formalism. It is a geometric property of phase space, and is analogous to the equivalence of all orthogonal co-ordinates systems in describing vectors.

* Poisson means "fish" in French