The work-energy theorem in a many-body system

This handout completes my discussion of the total potential energy of a many-body system in my 18 January lecture. I gave a prescription for the total potential energy that should be used in the work-energy theorem of such a system. But I didn't really justify it. This handout gives the justification. It is meant to complement the one given in the book.

We want to find how the work-energy theorem previously inferred for one particle applies to a many-particle system with many internal conservative forces $F_{\alpha\beta}$, such as gravitational forces. Applying our reasoning in previous lectures to the force on particle α, we know that the differential change of its kinetic energy dT_α is equal to the work done by the forces on α.

$$dT_\alpha = dW_\alpha = d\vec{r}_\alpha \cdot \vec{F}_\alpha$$

Since $\vec{F}_\alpha$ is the sum of internal forces $\vec{F}_{\alpha\beta}$ and external forces $\vec{F}_{\alpha,\mathrm{ext}}$ the work dW_α is likewise the sum of works due to these forces. This is true whether these forces are conservative or not.

Now we consider the overall change ΔT_α over many such infinitesimal intervals is the integral of dW_α, *i.e.,* $\Delta W_\alpha = \int d\vec{r}_\alpha F_\alpha$. However, we can't perform this integral explicitly since the forces involved depend on other co-ordinates r_β. That is ΔW_α depends on the path of motion. Thus there is no work-energy theorem for particle α in isolation.

We can recover a work-energy theorem only by considering the total kinetic energy of all the interacting particles, *i.e.,* $T = \sum_\alpha T_\alpha$. Now when we go to find the differential change in T we obtain contributions due to both $F_{\alpha\beta}$ and $F_{\beta\alpha}$. For now we focus on these internal forces and omit any external forces.

$$dT = \sum_\alpha dW_\alpha = \sum_{\alpha\beta} d\vec{r}_\alpha \cdot \vec{F}_{\alpha\beta}$$

We collect the two terms of this sum involving a given pair of particles α, β, so we include this pair only once in the sum by omitting the term where $\beta > \alpha$. Thus

$$dT = \sum_\alpha dW_\alpha = \sum_{\alpha<\beta} \left(d\vec{r}_\alpha \cdot \vec{F}_{\alpha\beta} + d\vec{r}_\beta \cdot \vec{F}_{\beta\alpha} \right).$$

Now the conservative feature of $F_{\alpha\beta}$ can be used to simplify: $\vec{F}_{\alpha\beta} = \vec{\nabla}_\alpha U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)$. Then the expression in (...) amounts to

$$\left(d\vec{r}_\alpha \cdot \vec{F}_{\alpha\beta} + d\vec{r}_\beta \cdot \vec{F}_{\beta\alpha} \right) = \left(d\vec{r}_\alpha \cdot \vec{\nabla}_\alpha U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta) + d\vec{r}_\beta \cdot \vec{\nabla}_\beta U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta) \right)$$

The right side is simply the total change in $U_{\alpha\beta}$ owing to any change in its two r's. That is,

$$\left(d\vec{r}_\alpha \cdot \vec{F}_{\alpha\beta} + d\vec{r}_\beta \cdot \vec{F}_{\beta\alpha} \right) = dU_{\alpha\beta}$$

Our expression for the differential change in T becomes simply

$$dT = \sum_{\alpha<\beta} dU_{\alpha\beta}$$

Now when we integrate both sides to obtain the net change in T, we obtain

$$\Delta T = \left(\sum_{\alpha<\beta} \Delta U_{\alpha\beta} \right)$$

Each of the $\Delta U_{\alpha\beta}$ on the right side is path independent; each depends only on its end points. Combining them, we conclude $\Delta T = \Delta U$ where

$$U = \sum_{\alpha<\beta} U_{\alpha\beta}$$

This justifies the statement I made in class, that the potential energy needed for the work-energy theorem is the sum contributions from different interactions, such as the gravitational interaction between particles 3 and 8. We sum over interactions, not particles.