Why the Action is Stationary: The Rubber-Band Analogy

This handout sets the stage for the most important principle in the course: Hamilton's Principle or "Principle of Least Action" It shows a connection between an arbitrary dynamical system and an anologous static system. The connection implies Hamilton'a Principle, and is a precursor to methods such as Feynmann path integrals[1] and the Chandler representation[2] of quantum fluctuations in a liquid.

Static counterpart of a dynamical system

We know that a system in equilibrium (like a hanging rope) comes to a state of extremal, stationary potential energy. The historical development of Hamilton's Principle was driven by efforts to generalize this static approach to treat moving systems. In this handout, I'll show you how a moving mechanical system may be converted to an equivalent fictitious static equilibrium system. The potential energy of this fictitious system is precisely the Action* S of the real one. Thus to minimize the Action of the real system is to minimize the potential energy of the fictitous one.

Now I'll define a fictitious system whose equilibrium state is constructed to correspond to a dynamical path. The system consists of an elastic cord or rubber band, as illustrated in Figure 1. The position of a point on the *unstretched* cord is labeled by s. It is the distance along the unstretched cord. It is convenient to think of the cord as consisting of a series of springs connected by nodes. We may label these nodes by an index i and we denote node i's position by $\tilde{r}_i$ †. Then the interval of s between successive nodes is denoted Δs.

In equilibrium, the forces on each node must balance. These consist of elastic forces from the springs, denoted f_{i+} and f_{i-} and external forces on the cord, denoted $f_{\rm non-el}$. The force balance at node i thus reads

$$\vec{f}_{i-} + \vec{f}_{i+} + \vec{f}_{\rm non-el} = 0$$

The spring force $\vec{f}_{i+}$ is proportional to the displacement between the spring ends. (We shall neglect the unstretched length for these springs, supposing that the springs are strongly stretched relative to their unstretched lengths.) Thus

$$\vec{f}_{i+} = \text{constant}(\tilde{r}_{i+1} - \tilde{r}_i)$$

Naturally the constant depends on how much elastic material is between two adjacent nodes. Indeed, if Δs were doubled, the stretching of the material would be half as much, and the force would be halved. Thus this constant must be inversely proportional to Δs; we denote it is $G/\Delta s$. On the other hand, the non-elastic force $f_{\rm non-el}$ must be proportional to the amount of material associated with the node: $\vec{f}_{\rm non-el} = \Delta s \tilde{F}_{non\ el}$, where $\tilde{F}$ is the

* Here I brazenly follow the colloquial usage of contemporary physicists rather than the careful historical usage of the text. By the Action I mean the time integral of kinetic minus potential energy.

† We use the ˜ mark to distinguish a point of the elastic system from the position $\vec{r}$ of a particle along a dynamical trajectory introduced below. $\tilde{r}$ is a vector, but we omit the vector sign for clarity

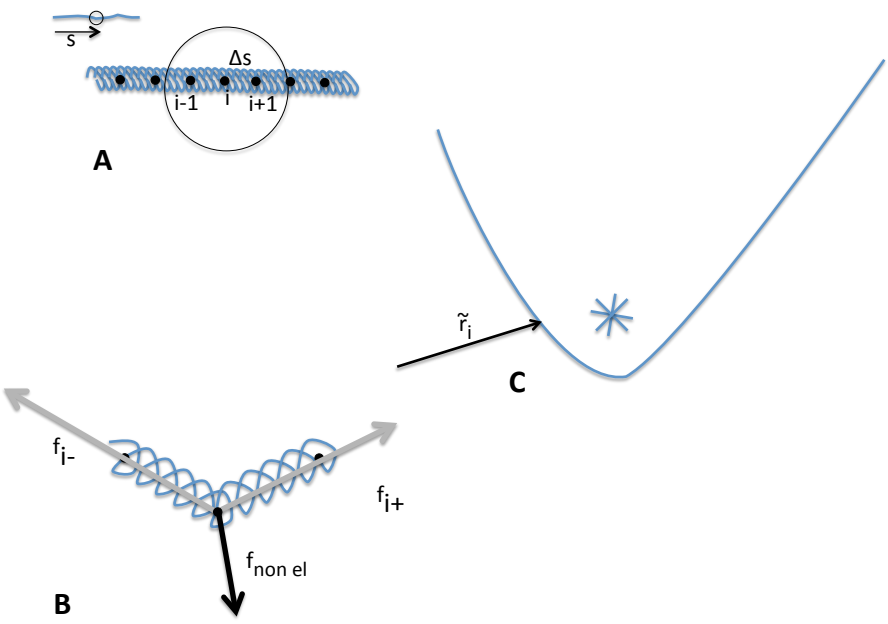

Figure 1: An elastic cord subject to an external force. A: a segment of the unstretched cord in its resting configuration; close-up shows the nodes i and the (very small) unstretched length Δs between adjacent nodes. B: forces on node i with external forces acting, including the force f_{i+} from the next spring in the series, the force f_{i-} from the preceding spring, and the force $f_{\rm non-el}$ from the external source. The angle between springs is exaggerated for clarity. C: the path of the cord under some external force (a repulsive charge). The position of a given node i is indicated by $\tilde{r}_i$.

force per unit of material. Combining these facts and moving the $f_{\rm non-el}$ part to the right hand side, we infer

$$\frac{G}{\Delta s}\left[(\tilde{r}_{i+1}-\tilde{r}_i)-(\tilde{r}_i-\tilde{r}_{i-1})\right]=-\Delta s\tilde{F}_{\rm non-el}$$

As we approach the limit of a large number of nodes, these differences become derivitives, and our equation becomes

$$\frac{G}{\Delta s}\frac{d}{di}\frac{d\tilde{r}}{di}=-\Delta s\tilde{F}_{\rm non-el}$$

We may now use the original parameter s, noting that $s=i\Delta s$, so that $ds=\Delta s\ di$. Converting both di's into ds, we find

$$G\frac{d^2\tilde{r}}{ds^2}=-\tilde{F}_{\rm non-el}.$$

We see that our fictitious elastic equilibrium system has a path $\tilde{r}$ governed by an equation identical to Newton's equation for a particle

- whose mass is G,
- whose time is s,
- whose position at time s is $\tilde{r}(s)$, and
- whose force is *minus* $\tilde{F}_{\rm non-el}$.

If we follow this dictionary, any dynamical trajectory of a particle may be viewed as the path of the corresponding elastic cord. This same trick is not limited to single-particle systems. If we have several particles, we simply create an elastic cord for each one and proceed as above. The only peculiar thing about this correspondence is the reversal of the external force. This reversal arose because in the elastic system, all forces must sum to zero, so the elastic forces equal the *negative* of the non-elastic force. This reversal is natural if we look at specific cases such as Figure 1C. Here we have a uniformly charged cord being pushed away from a point charge of the same sign: in this case the cord stretches around the back side of the charge. According to the derivation above, this picture should describe a trajectory of a massive point charge around an *attractive* external charge. And indeed, we know the trajectory of such a particle—like the trajectory of a planet—should follow a (hyperbolic) curve around the back of the external point*. This picture adds intuitive confirmation to the derivation.

The idea of relating the dynamical system to a fictitious system in equilibrium is known as D'Alembert's principle. This is the historical forerunner of Hamilton's Principle and it is treated in many text books, including *Classical Mechanics* by Herbert Goldstein.

Variational prescription for the dynamical trajectory

The advantage of our elastic system is that we can immediately give a variational prescription—*i.e.,* an optimization formulation—for the equilibrium path $\tilde{r}(s)$: it is the path for which the total potential energy $\mathcal{U}[\tilde{r}(s)]$ is stationary. (Typically it is minimal.) Here we suppose that the non-elastic force is conservative, so that the non-elastic force on a segment Δs $\tilde{F}_{\rm non-el} = (\vec{f}_{\rm non-el})_i = -\vec{\nabla} U_i$. Thus $\tilde{F}_i$ is also the negative gradient of $\tilde{U} = U_i/\Delta s$. We may now write the total elastic energy by summing the contributions from all the segments i

$$\mathcal{U} = \mathcal{U}_{el} + \mathcal{U}_{\rm non-el}$$

$$\mathcal{U}_{\rm non-el} = \sum_i U_i = \sum_i \Delta s (U_i/\Delta s) \to \int ds\ \tilde{U}$$

Likewise we may write $\mathcal{U}_{el}$ as the sum of contributions $\frac{1}{2}(G/\Delta s)(\tilde{r}_{i+1} - \tilde{r}_i)^2$ from each spring:

$$\mathcal{U}_{el} = \sum_i \frac{\frac{1}{2}G}{\Delta s} \left(\frac{dr}{di}\right)^2 = \sum_i \Delta s\ \frac{\frac{1}{2}G}{\Delta s^2} \left(\frac{d\tilde{r}}{di}\right)^2$$

as before, we can use $di\Delta s = ds$ to absorb the Δs factors, thus obtaining

$$= \sum_i \Delta s\ \frac{1}{2} G \left(\frac{d\tilde{r}}{ds}\right)^2 \to \int ds\ \frac{1}{2} G \left(\frac{d\tilde{r}}{ds}\right)^2$$

* For both the static cord and the moving particle another trajectory with the same end-points, which stays above the external charge, is possible.

Combining these parts, we find a familiar-looking expression for $\mathcal{U}$:

$$\mathcal{U}[\tilde{r}] = \int ds \left[\frac{1}{2} G \left(\frac{d\tilde{r}}{ds} \right)^2 + \tilde{U} \right]$$

Because this is the potential energy of an equilibrium elastic system, it is necessarily stationary for the actual path. On the other hand, we may interpret the integral in terms of the dynamical quantities, substituting m for G, $\vec{r}(t)$ for $\tilde{r}(s)$, and dt for ds. We also recall that the counterpart of the force $\tilde{F}_{\rm non-el}$ for the nonelastic force must be replaced by $-\vec{F}$, the negative of the particle force. Since $\tilde{F}$ has potential energy $\tilde{U}$, its negative $\vec{F}$ has potential energy $U = -\tilde{U}$. Combining, we get

$$\mathcal{U} = \int dt \ \left[\frac{1}{2} m \left(\frac{d\vec{r}}{dt} \right)^2 - U \right]$$

We recognize the first term in the integrand as the kinetic energy T. We have thus shown that this integral is stationary when the trajectory $\vec{r}(t)$ is the actual dynamical motion. The functional $\int dt(T-U)$ is known as the Action and is denoted by the letter S. The integrand $T-U$ is denoted by $\mathcal{L}$ and is called the Lagrangian.

In Chapters 6 and 7, you will see a more conventional explanation of Hamilton's Principle. The complementary approach above offers a more intuitive way to understand it.

References

[1] `http://en.wikipedia.org/wiki/Path_integral_formulation`

[2] CHANDLER, D (CHANDLER, D); WOLYNES, PG (WOLYNES, PG) Source: JOURNAL OF CHEMICAL PHYSICS Volume: 74 Issue: 7 Pages: 4078-4095 DOI: 10.1063/1.441588 Published: 1981