Problem 2.1 (3 points) Light rocket

The momentum p of a photon is proportional to its energy E, unlike that of a massive particle. The proportionality constant is the speed of light c: thus $p = E/c$. A light rocket works by shining a light beam out the back end of the rocket. A certain light rocket gives one Newton of thrust. What is the power of the light beam in watts?

Problem 2.2 (10 points) Underwater SuperSoaker

The SuperSoaker water gun works by pumping compressed air into a tank mostly filled with water, and then releasing the water through a nozzle at a speed v_{ex}, thus causing a reaction force. Suppose that the gun is operated under water, so that there is also a water resistance force $\vec{D} = -b\,\vec{v}$ (presumed to be linear).

a) Express the reaction force F_R in terms of v_{ex} and (constant) rate of mass loss $\dot{m}$.

b) After squirting for a while, the gun's original mass m_0 has decreased to m and the speed has increased from zero to v. Use the equation of motion to find this v in terms of m/m_0, and v_{ex}

c) The CPS 2000 has an initial mass of 4 kg, an output of about 1 kg/sec, and a muzzle velocity of about 10 meters/sec. The linear drag coefficient b for this size object in water is about .01 Newtons/(meter per sec) or .02 kg/sec. How much mass needs to be expelled for the soaker to attain a speed of 1 meter/sec?

d) Does the drag on the outgoing stream of water matter?

Problem 2.3 (7 points) Accreting asteroid

Problem 3.29 in the text.

Problem 2.4 (10 points) Square vs Conservative

A force field is described by a smooth function $\vec{F}(\vec{r})$ in the plane. We consider this function near a given $\vec{r}_0$ which we take as the origin. As with any smooth function, $\vec{F}$ near this origin is a linear function to arbitrarily good accuracy. Thus we may write

$$\vec{F}(x, y) = \vec{F}_0 + (a\;x + b\;y)\hat{x} + (c\;x + e\;y)\hat{y}$$

Now consider a rectangular closed path that starts from the origin, moves a small distance Δy in the positive y direction, then a small distance Δx in the positive x direction, then reverses the Δy motion and finally reverses the Δx motion.

a) Find the work W done by F in traversing this path. Evidently if $W \neq 0$, this force is not conservative

b) What conditions on F_0, a, b, c, and e guarantee that $W = 0$?

c) Compare this condition with Eq. (4.36) in the text. Are the two conditions equivalent?

Solution:

2.1 ....

Momentum Δp in time Δt is $F\Delta t$. The corresponding energy ΔE emitted is $(dE/dt)\ \Delta t$. So power $dE/dt = Fc = 1 \times (3 \times 10^8) = 300$ megawatts.

note: the rocket would then be losing mass, according to $E = mc^2$ at a rate $\dot{m} = \dot{E}/c^2 = 300 = Fc/c^2 = .3 \times 10^{-8}$ kg/sec . For a given amount of thrust, emitting light is less costly in mass than emitting massive particles.

2.2 ....

a) $F_R = dp/dt = v_{ex}\ \dot{m}$

b)

$$m(t)\dot{v} = F_R - D = v_{ex}\dot{m} - bv$$

$$\frac{\dot{v}}{v_{ex}\dot{m} - bv} = 1/m(t)$$

$$\int_0^v \frac{dv'}{v_{ex}\dot{m} - bv'} = \int_0^t \frac{dt}{m(t')} = \int_{m_0}^m \frac{-dm'}{\dot{m}\ m'} = \frac{-1}{\dot{m}} \log \frac{m}{m_0}$$

The integral on the left is

$$\int_0^v \frac{dv'}{v_{ex}\dot{m} - bv'} = -\frac{1}{b} \log \frac{v_{ex}\dot{m} - bv}{v_{ex}\dot{m}} = -\frac{1}{b} \log\left(1 - bv/(v_{ex}\dot{m})\right)$$

Combining,

$$\log\left(1 - bv/(v_{ex}\dot{m})\right) = \frac{b}{\dot{m}} \log \frac{m}{m_0} = \log\left(\frac{m}{m_0}\right)^{b/\dot{m}}$$

so that

$$1 - bv/(v_{ex}\dot{m}) = \left(\frac{m}{m_0}\right)^{b/\dot{m}}$$

or

$$v = v_{ex}\left(\frac{\dot{m}}{b}\right)\left(1 - \left(\frac{m}{m_0}\right)^{b/\dot{m}}\right) \tag{1}$$

c) $b/\dot{m} = .01$; $v/v_{ex} = .1$; $v/v_{ex}(b/\dot{m}) = .001$. Since the left side of (1) is small, m must be near m_0. Expanding $m = m_0 - \Delta m$ for small Δm, gives $(\frac{m}{m_0})^{b/\dot{m}} = 1 - (b/\dot{m})\Delta m/m_0 + \ldots.$
so (1) becomes

$$\frac{v}{v_{ex}}\left(\frac{b}{\dot{m}}\right) \simeq \left(\frac{b}{\dot{m}}\right)\frac{\Delta m}{m_0} + \ldots$$

Thus $\Delta m = m_0(v/v_{ex}) = .4$ kg. The drag force doesn't matter in this regime. Still, for such large velocities our linear approximation of D is suspect. Probably the needed mass is greater than this calculation gives.

d) The drag on the outgoing stream of water doesn't matter. What matters for the rocket's acceleration is its momentum when it leaves the rocket. The later fate of that water is immaterial.

2.3 ....

$$I_0\omega_0 = L_0 = L_f = I_f\omega_f$$

so

$$\omega_f = \omega_0(I_0/I_f) = \omega_0\left(\frac{\frac{2}{5}M_0R_0^2}{\frac{2}{5}M_fR_f^2}\right) = \omega_0\left(\frac{M_0R_0^2}{(8M_0)(2R_0)^2}\right) = \omega_0/32$$

2.4 ....

a) We first consider the work done by $\vec{F}_0$.

$$W_0 = \vec{F}_0 \cdot (\hat{y}\Delta y + \hat{x}\Delta x - \hat{y}\Delta y - \hat{x}\Delta x) = 0$$

naturally. Now the linear part of $\vec{F}$ contributes work W as follows

$$W = \int_0^{\Delta y} c\ y\ dy + \int_0^{\Delta x} (ax + b\Delta y)\ dx + \int_{\Delta y}^{0} (c\Delta x + ey)\ dy + \int_{\Delta x}^{0} (ax)\ dx$$

$$= \int_0^{\Delta y} c\ y\ dy + \int_0^{\Delta x} (ax + b\Delta y)\ dx - \int_0^{\Delta y} (c\Delta x + ey)\ dy - \int_0^{\Delta x} (ax)\ dx$$

Performing the constant parts of the integrals

$$W = \int_0^{\Delta y} c\ y\ dy + \int_0^{\Delta x} (ax)dx + b\Delta y\Delta x - (c\Delta x)\Delta y - \int_0^{\Delta y} (ey)\ dy - \int_0^{\Delta x} (ax)\ dx$$

The remaining integrals cancel pairwise so that $W = (b-c)\Delta x\Delta y$. Thus the condition for $\vec{F}$ to be conservative is $c = b$.

b)

$$\nabla \times \vec{F} = \partial_x F_y - \partial_y F_x = c - b$$

The condition for the curl to vanish is $c = b$ as in a). Thus they are equivalent. .