Problem 3.1 (12 points) Chain of springs

A spring chain is formed by connecting a large number N of identical massless springs in sequence, with small masses m at the end of each spring. The end of the first spring is attached to the ceiling, leaving the others to hang down and reach an equilibrium configuration of lowest potential energy. Each mass i then has a position y_i below the ceiling. The gravitational potential energy of mass i is $m\ g\ y_i$. The ith spring as a potential energy $\frac{1}{2}k\ (y_i - y_{i-1})^2$. (These springs have a negligible unstretched length.)

a) Express the total energy U as a sum on i.

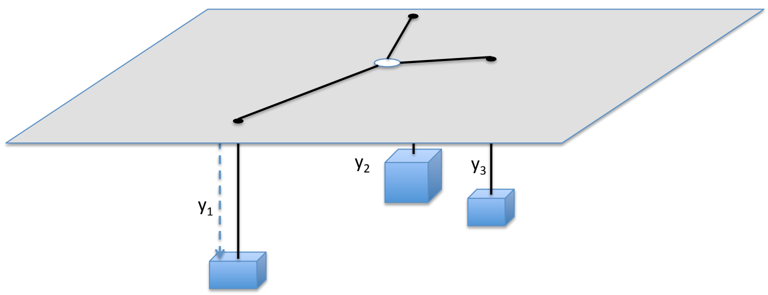
b) Suppose that N is very large so that $\sum_i$ can be replaced by $\int di$ and $(y_i - y_{i-1})$ can be replaced by dy/di or y'. Express $U[y]$ as an integral over i

c) Qualitatively, how does the stretch of the springs vary with position i?

d) Find a differential equation satisfied by the stationary $y(i)$ (the Euler Lagrange equation).

e) What is the length of the hanging chain, *i.e.,* what is the value of $y(N)$? How does it vary with N for fixed total mass Nm?

Problem 3.2 (7 points) Fancy constraint



A massless disk slides on a frictionless table. It is attached to three weights by inextensible strings as shown. The disk is not necessarily in static equilibrium; it may be accelerating. If the disk moves, the heights of the weights $y_1 ... y_3$ change. But clearly these three y's cannot vary independently, since the disk can only move in two dimensions. Thus there is a constraint condition that limits how the y's can change. If it moves slightly from the location shown, this constraint has the form

$$dy_1 + .2dy_2 + .3dy_3 = 0 \tag{1}$$

This constraint is enforced by tensions T_1, T_2, T_3 in the three strings. Find a condition on T_2 and T_3 in terms of T_1 such that no net work is done by these tensions for motion consistent with the constraint (1).

Problem 3.2 (10 points) Fancy variation

We wish to find a stationary path $y(x)$ for the functional

$$S[y] \equiv \int_0^1 dx f(y, y'')$$

with $y(0)$, $y(1)$, $y'(0)$, and $y'(1)$ all fixed. Notice that this differs from the $S[y]$ treated in the text: the integrand now depends on the *second* derivative $y(x)$ rather than the first. Such variational problems arise *eg.* in hydrodynamics. Happily, one may find a generalized Euler-Lagrange equation for the stationary path using an argument closely analogous to that of Section 6.2. Derive this Euler Lagrange equation.

Solution:

3.1 ....

a)

$$U = \sum_{i=1}^{N} \left[mgy_i + \frac{1}{2} k (y_i - y_{i-1})^2 \right]$$

b)

$$U = \int_0^N di \ \left[mgy(i) + \frac{1}{2} k (y')^2 \right]$$

c) the greatest stretch is at the top, since the whole mass Nm is pulling on the top spring; the smallest stretch is at the bottom, and it is N times smaller than at the top. When N is large, the stretch at the bottom is arbitrarily small compared to that at the top. We can take it to be zero.

d) denote integrand in [...] by $f(y, y')$. Then

$$\partial f / \partial y = mg$$

$$\partial f / \partial y' = ky'$$

$$\frac{d}{di} \frac{\partial f}{\partial y'} = ky''$$

So Euler Lagrange equation reads

$$0 = -\frac{\partial f}{\partial y} + \frac{d}{di} \frac{\partial f}{\partial y'} = -mg + ky'' = 0$$

e) This is the equation for free fall in upward-pointing gravity

$$y(i) = y_0 + v \ i + \frac{1}{2} (gm/k) \ i^2$$

To find $y(N)$, we need to know the free parameters $y(0)$ and v. We let the ceiling be at $y(0) = 0$. To find v, we use the knowledge that $y' \to 0$ for $i \to N$. Now,

$$y'(i) = v + (gm/k)i;$$

so

$$0 = y'(N) = v + (gm/k)N, \quad \text{so that} \quad v = -(gm/k)N$$

Substituting into the $y(i)$ formula, we get

$$y(N) = \frac{1}{2} (gm/k) \ N^2 \quad = \frac{1}{2} (gNm/k) N$$

For fixed total mass mM the stretching is proportional to the number of segments.

3.2 ....

$dW = T_1 dy_1 + T_2 dy_2 + T_3 dy_3$ For motion consistent with the constraints.

$$dy_1 = -.2 dy_2 - .3 dy_3$$

so that

$$dW = T_1(-.2dy_2 - .3dy_3) + T_2 \ dy_2 + T_3 \ dy_3 = (T_2 - .2T_1) \ dy_2 + (T_3 - .3T_1) \ dy_3$$

All these motions dy_2, dy_3 are consistent with the constraints, so dW can only be zero if the two (...) factors are zero; *i.e.,*

$$T_2 = .2T_1; \quad T_3 = .3T_1$$

3.3 ....

As in the chapter, we let the varied path $Y(x) = y(x) + \alpha\eta(x)$, where $y(x)$ is the stationary path sought. As before we wish to find the path $y(x)$ such that

$$S(\alpha) \equiv_0^1 dx f(Y, Y'')$$

is stationary, *i.e.*,

$$0 = \frac{dS}{d\alpha} = \int_0^1 dx\ \eta \frac{\partial f}{\partial y} + \eta'' \frac{\partial f}{\partial y''}$$

As before we can express the integral without η'' using integration by parts

$$\int_0^1 \eta'' g(x) = -\int_0^1 \eta' g'(x) = +\int_0^1 \eta g''(x)$$

The endpoint contributions in the integrations by parts vanish because, by hypothesis $\eta(0)$ and $\eta(1)$ are zero, as are $\eta'(0)$ and $\eta'(1)$. Here $g(x)$ is $\partial f/\partial y''$ and g'' means $(d^2/dx^2)g$. Thus the stationary condition can be written as

$$0 = \frac{dS}{d\alpha} = \int_0^1 dx\ \eta(x) \left[\frac{\partial f}{\partial y} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}\right]$$

Since the integral must vanish for any choice of η, the expression in [...] must also vanish, *i.e.*,

$$\frac{\partial f}{\partial y} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$

This is our generalized Euler Lagrange equation.