5.1 (8 points) Uniform magnetic field.

A uniform magnetic field B in the $\hat{z}$ direction has a vector potential in cylindrical co-ordinates (ρ, ϕ, z) given by $A = \frac{1}{2}B\ \rho\hat{\phi}$.

a) Using the Lagrangian of Eq. 7.103, find the equations of motion for the radial co-ordinate ρ and the angular coordinate ϕ. What is the condition that ρ remain constant?

5.2 (8 points) wobbly orbit

The earth's orbit is nearly circular, but not quite. The radial motion oscillates in a narrow range around the minimum of the effective potential $U_{\rm eff}$. We call this radius r^*. If this range is narrow enough, $U_{\rm eff}$ may be approximated as $A(r - r^*)^2 - B$, so the radial motion is simple harmonic motion.

a) Find the coefficient A in terms of the strength of the gravitational potential Gm_1m_2/r^*.
b) Find the period of the simple harmonic radial motion.
c) Compare with the angular revolution time $2\pi/\dot{\phi}$

5.3 (12 points) virial theorem.

A particle of mass m moves in a central potential $U(r)$ of the form $U(r) = Ar^a$, where the exponent a may be a positive or negative number. We have seen that the effective potential $U_{\rm eff}(r)$ for central force motion has the form $U(r) + B/r^2$, where B is a positive constant depending on the angular momentum. In the gravitational case, $a = -1$, and A is negative. Then $U_{\rm eff}$ has a minimum at at some finite, nonzero value of r. If A is positive, there is no minimum.

a) If A is positive, what is the range of a such that $U_{\rm eff}$ does have a minimum (at finite, nonzero r)? What is the range of a that gives a minimum (not a maximum) if A is negative?

The quantity $G \equiv \vec{r} \cdot \vec{p}$ is called the "virial of Clausius." As usual, the momentum $\vec{p} = m\dot{\vec{r}}$. This quantity is interesting because its time derivative $\dot{G}$ is a fixed linear combination of T and U whenever U has the form Ar^a. That is, $\dot{G} = C\ T + D\ U$, where C and D are constants.

b) Find the constants C and D in terms of a.
c) Now suppose that $U_{\rm eff}$ has a minimum and that the motion is bounded between some $r_{\rm min}$ and $r_{\rm max}$. As we know, the particle moves periodically in r with a period τ. The change of G over a period must be zero. Use this fact to find a relation between the time average of T and the time average of U over a period. This relation depends only on the exponent a.

Solution:

5.1 ______________________________

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + q\dot{\vec{r}} \cdot \vec{A}$$

In the polar co-ordinates of interest

$$\mathcal{L} = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + q\rho\dot{\phi}(\frac{1}{2}B\rho) = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}\rho^2(m\dot{\phi}^2 + qB\dot{\phi})$$

For the ϕ co-ordinate we have

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0; \quad \frac{\partial\mathcal{L}}{\partial\dot\phi} = m\rho^2\dot\phi + \frac{1}{2}qB\rho^2 = \rho^2(m\dot\phi + \frac{1}{2}qB)$$

So that $\rho^2(m\dot\phi + \frac{1}{2}qB) = \text{constant}$ For the ρ variable we have

$$\frac{\partial\mathcal{L}}{\partial\rho} = \rho(m\dot\phi^2 + qB\dot\phi); \quad \frac{\partial\mathcal{L}}{\partial\dot\rho} = m\dot\rho$$

so that

$$m\ddot\rho = \rho(m\dot\phi^2 + qB\dot\phi) = \rho\dot\phi\ (m\dot\phi + qB) \tag{1}$$

Now we look for conditions in which ρ is constant, *i.e.,* the orbit is circular. Since $\ddot\rho$ must be zero, (1) implies $m\dot\phi = -qB$. The angular frequency is fixed at $-qB/m$. This is called the cyclotron frequency. This condition doesn't fix the value of ρ itself. Indeed, the kinetic energy $\frac{1}{2}\rho^2 m\dot\phi^2 = \rho^2 q^2 B^2/m$ can only change if ρ changes: particles with different kinetic energies have different orbital radii.

Given an initial velocity $\vec v$ one can always place the origin at a distance ρ from the particle such that $\vec v = \rho\dot\phi\ \hat\phi$, and $\dot\phi = -B/m$. As shown above, the subsequent motion will have fixed ρ and thus be a circular orbit. Thus all orbits are circular orbits around some center with angular velocity $\omega = qB/m$.

5.2

$$U_{\text{eff}} = \frac{\ell^2}{2m\ r^2} - GmM/r$$

At $r = r^*$, $0 = dU_{\text{eff}}/dr = -\frac{\ell^2}{mr^3} + GmM/r^2$ or $0 = -\ell^2/m + GmM\ r^*$. Then

$$U_{\text{eff}}(r^*) = \frac{1}{2}GmM\ r^*/r^{*2} - GmM/r^* = -\frac{1}{2}GmM/r^*$$

The second derivative at r^* is given by

$$\frac{d}{dr}\frac{dU_{\text{eff}}}{dr} = \frac{3\ell^2}{mr^4} - 2\frac{GmM}{r^3} = 3\frac{GmM\ r^*}{r^{*4}} - 2\frac{GmM}{r^{*3}} = GmM/r^{*3} = |U(r^*)|/r^{*2}$$

Thus

$$U_{\text{eff}} = U_{\text{eff}}(r^*) + \frac{1}{2}\frac{d^2U_{\text{eff}}}{dr^2}(r-r^*)^2 = U_{\text{eff}}(r^*) + \frac{1}{2}|U(r^*)|(r-r^*)^2/r^{*2}$$

b) Let $u = r - r^*$ then $m\ddot u = -dU_{\text{eff}}/dr = -|U(r^*)|u/r^{*2}$. But in a harmonic oscillator $\ddot u = -\omega^2\ u$. Therefore in this oscillator $\omega^2 = |U(r^*)|/(mr^{*2})$. radial period $\tau = 2\pi/\omega = 2\pi\sqrt{mr^{*2}/|U(r^*)|} = 2\pi\sqrt{r^{*3}/GM}$

c) The $\dot\phi$ for a nearly circular orbit is determined by radial acceleration: $mr^*\dot\phi^2 = F = -dU/dr = U(r^*)/r^*$. So

$$2\pi/\dot\phi \to 2\pi\sqrt{\frac{mr^{*2}}{U(r^*)}} = 2\pi\sqrt{\frac{mr^{*2}}{GmM/r^*}} = 2\pi\sqrt{\frac{r^{*3}}{GM}}$$

The two periods match, so the motion is a closed orbit as we anticipated.

5.3

a) If there is a minimum for some r, then $0 = dU/dr = -2B/r^3 + aAr^{a-1}$, so $2B/r^2 = aA\ r^a$. If A is positive, a must also be positive. We have unique extremum of U for any positive a. For positive A and a, $U \to +\infty$ as $r \to 0$ or $r \to \infty$. Thus the single extremum must be a minimum. If A is negative, a must also be negative, $(2B/aA) = r^{a+2}$. If $-2 < a < 0$ then the negative A term dominates at large r and U_{eff} has a minimum. But if $a < -2$ then the B term dominates at large r, and the negative A term dominates at small r, so U_{eff} has a maximum rather than a minimum.

b)

$$\dot{G} = \dot{\vec{r}} \cdot \vec{p} + \vec{r} \cdot \dot{\vec{p}} = m\dot{\vec{r}} \cdot \dot{\vec{r}} + \vec{r} \cdot F = 2T - rdU/dr = 2T - aU$$

so $C = 2$ and $D = -a$.

c)

$$0 = G(\tau) - G(0) = \int_0^\tau dt\ \dot{G} = \int_0^\tau dt(2T - aU) = \tau(2\langle T\rangle - a\langle U\rangle)$$

where $\langle ... \rangle$ indicates the average over one period.