6.1 (12 points) dilation co-ordinate.

We have seen that invariance of a system under spatial transformations like translation or rotation lead to conserved quantities. Thus if $\mathcal{L}$ is unchanged under a rotation by ϕ about some axis, then it follows that $\partial\mathcal{L}/\partial\dot{\phi}$ is a constant of the motion. More generally, if we know how $\mathcal{L}$ changes under a spatial operation we gain information about the time derivative of its generalized momentum. In this problem we consider the operation of spatial dilation: $\vec{r} \to \tilde{r} \equiv \vec{r}/\lambda$. This is interesting because some systems behave in a simple way under dilation. A major example is a power-law potential $U(r) = Ar^a$. We proceed as usual by taking the dilation variable λ as a generalized co-ordinate for our system.

a) Using the $\tilde{r}$ and λ variables, find $\partial U/\partial\lambda$ at $\lambda = 1$. You see that the derivative is a simple multiple of U itself. This property of any power-law function is known as *homogeneity.*

b) Again using the co-ordinates $\tilde{r}$ and λ, find $\partial\mathcal{L}/\partial\lambda$ evaluated at $\lambda = 1$ and $\dot{\lambda} = 0$. It is a linear combination of T and U, as in the previous problem.

c) Compute $\partial\mathcal{L}/\partial\dot{\lambda}$, at $\lambda = 1$ and $\dot{\lambda} = 0$. This is the generalized momentum for dilation. Compare with G of virial theorem problem on the last problem set.

This dilation operation generalizes to a system of many particles. If the particles interact pairwise with a power-law potential $U_{ij} = A_{ij}|r_i - r_j|^a$, then $\partial\mathcal{L}/\partial\lambda$ is again a linear combination of the total kinetic and potential potential energies. This leads to a powerful generalization of the virial theorem.

6.2 (6 points) disappearing sun

One day half the mass of the sun suddenly disappears. This changes the Earth's (initially circular) orbit. What is its final mechanical energy E? What is the shape of the orbit? The virial theorem is useful for this problem.

6.3 (15 points) best boost

A satellite is to be boosted from a circular orbit of radius R and speed v to a bigger one of radius λR by firing rockets. For simplicity we will assume that the rockets act over a time much shorter than any orbital period. An initial boost sends the satellite into a "transfer orbit" that will connect the initial R circle to the desired λR circle. The rocket chemical energy required is a fixed constant times the change in speed Δv. (We suppose that the mass expelled from the rocket is much smaller than the mass of the satellite.)

a) By what factor must the mechanical energy E change during the initial boost? Note that the orbit's major axis after this boost will have its final value.

b) A second boost is needed in order to change the transfer orbit into the final circular orbit. This boost requires a second change of speed $\Delta_2 v$. What is the ratio of this second Δv to the first one?

c) A second way to accomplish the boost is to apply this procedure twice to produce two $\sqrt{\lambda}$ increases in R. How much rocket energy does this use relative to the first method?

Solution:

6.1

a) $\partial U(r)/\partial\lambda = \partial U(\lambda\tilde{r})/\partial\lambda = \tilde{r}\ dU(\lambda\tilde{r})/d(\lambda\tilde{r}) = a\ Ar^a = aU$.

b)

$$T = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m(\frac{d}{dt}(\lambda\tilde{r}))^2 = \frac{1}{2}m(\dot{\lambda}\tilde{r} + \lambda\dot{\tilde{r}})^2 = \frac{1}{2}m\left(\lambda^2\dot{\tilde{r}}^2 + 2\lambda\dot{\lambda}\tilde{r}\cdot\dot{\tilde{r}} + \dot{\lambda}^2\tilde{r}^2\right)$$

$$\frac{\partial T}{\partial\lambda} = m\lambda\dot{\tilde{r}}^2 + m\dot{\lambda}\tilde{r}\cdot\dot{\tilde{r}}$$

At $\lambda = 1; \dot{\lambda} = 0$ this reduces to

$$\frac{\partial T}{\partial\lambda} = 2T$$

Thus

$$\frac{\partial\mathcal{L}}{\partial\lambda} = 2T - aU$$

c)

$$\frac{\partial\mathcal{L}}{\partial\dot{\lambda}} = \frac{\partial T}{\partial\dot{\lambda}} = m\lambda\tilde{r}\cdot\dot{\tilde{r}} + m\dot{\lambda}\tilde{r}^2$$

At $\lambda = 1; \dot{\lambda} = 0$ this reduces to

$$\frac{\partial\mathcal{L}}{\partial\dot{\lambda}} = mr\cdot\dot{r} = r\cdot p$$

It is just the G of the previous problem. Thus G is the generalized momentum corresponding to a spatial dilation.

6.2

For a circular orbit the virial theorem says that the potential energy is twice the kinetic energy. If the mass of the sun is suddenly halved, the potential energy $-GmM/r$ is halved in magnitude, while the kinetic energy remains unchanged. Thus the potential energy is no longer twice the kinetic energy—the two are now equal and opposite. Thus the total energy E is 0, and the earth is in a marginal parabolic escape orbit.

6.3

a) In class it was shown that the diameter d of the orbit depends only on the energy E, and not the angular momentum ℓ. We may find the quantitative dependence by taking any orbit we like. We take one that is at rest at the apogee $r = r_+$. Then $E = -GmM/r_+ = -GmM/d$. The orbit before the boost has diameter $2R$. The orbit after the boost will have $r_- = R$ and $r_+ = \lambda R$. Thus its diameter $d = r_+ + r_- = (1+\lambda)R$. The ratio of energies E_t/E_0 is evidently $2/(1+\lambda)$.

b) We first find the initial Δv_0 needed for the given λ. Since the potential energy doesn't change during the boost, $E_t - E_0 = \Delta T = \frac{1}{2}m[(v_0 + \Delta v)^2 - {v_0}^2]$. From the virial theorem $T_0 = -\frac{1}{2}U_0 = -E_0$. Thus

$$E_t/E_0 - 1 = -[(1 + \Delta v/v_0)^2 - 1] = -2\Delta v/v_0 - (\Delta v/v_0)^2$$

Or using the quadratic formula

$$\frac{\Delta v}{v_0} = \frac{2 \pm \sqrt{4 + 4(1 - E_t/E_0)}}{-2} = -1 \pm \sqrt{1 + (1 - E_t/E_0)}$$

Evidently one can achieve the needed final energy by a (larger) negative Δv, but we ignore that option.

$$\frac{\Delta v}{v_0} = -1 + \sqrt{2 - 2/(1+\lambda)} = -1 + \sqrt{2\lambda/(1+\lambda)}$$

The formula gives $\Delta v = 0$ for $\lambda = 1$ as it must.

The second boost must give the orbit the energy E_f of a circular orbit: $E_f = -GmM/(2\lambda R)$. So that $E_f/E_t = (1+\lambda)/(2\lambda)$. Again the change of energy $E_f - E_t$ is entirely due to the kinetic energy added by the boost.

$$E_f - E_t = \frac{1}{2}m[{v_f}^2 - (v_f - \Delta_2 v)^2] = \frac{1}{2}m{v_f}^2[1 - (1 - \Delta_2 v/v_f)^2]$$

Since the final orbit is circular, $\frac{1}{2}m{v_f}^2 = -E_f$ as before.

$$1 - E_t/E_f = -[1 - (1 - \Delta_2 v/v_f)^2] = -2(\Delta_2 v/v_f) + (\Delta_2 v/v_f)^2$$

Then

$$\frac{\Delta_2 v}{v_f} = \frac{2 \pm \sqrt{4 + 4(1 - E_t/E_f)}}{2} = 1 \pm \sqrt{2 - E_t/E_f}$$

Keeping only the smaller root as before,

$$\frac{\Delta_2 v}{v_f} = 1 - \sqrt{2 - (2\lambda)/(1+\lambda)} = 1 - \sqrt{2/(1+\lambda)}$$

We now find the requested ratio

$$\frac{\Delta_2 v}{\Delta v} = \frac{v_f}{v_0} \; \frac{1 - \sqrt{2/(1+\lambda)}}{-1 + \sqrt{2\lambda/(1+\lambda)}}$$

We note that $v_f/v_0 = \sqrt{T_f/T_0} = \sqrt{E_f/E_0} = 1/\sqrt{\lambda}$. So,

$$\frac{\Delta_2 v}{\Delta v} = \frac{1}{\sqrt{\lambda}} \; \frac{1 - \sqrt{2/(1+\lambda)}}{-1 + \sqrt{2\lambda/(1+\lambda)}} = \frac{1}{\sqrt{\lambda}} \; \frac{\sqrt{1+\lambda} - \sqrt{2}}{\sqrt{2\lambda} - \sqrt{1+\lambda}}$$

This ratio is unity for λ near 1, and falls off as $1/\sqrt{\lambda}$ for large λ.

c) *This part was complicated and it required insights that would not be apparent from the book or the lecture. It is harder than you should be expected to do. Still, the solution below is instructive.*

It is more convenient to express the Δv's in terms of the initial v_0:

$$\frac{\Delta v}{v_0} + \frac{\Delta_2 v}{v_0} = -1 + \sqrt{2\lambda/(1+\lambda)} + \frac{v_f}{v_0} \; (1 - \sqrt{2/(1+\lambda)})$$

$$= -1 + \sqrt{2\lambda/(1+\lambda)} + \frac{1}{\sqrt{\lambda}}\,(1 - \sqrt{2/(1+\lambda)})$$

We define this function as $f(\lambda)$. In the two stage process we have an intermediate circular orbit with speed v_m. We may evidently achieve the given final orbit by making two successive operations like a) and b), each with expansion factor $\sqrt{\lambda}$. Here the first pair of Δv's is given by

$$\Delta v + \Delta_2 v = v_0\ f(\sqrt{\lambda})$$

The second pair $\Delta' v, \Delta_2' v$ is the same as the first pair, except that its initial speed is v_m instead of v_0:

$$\Delta' v + \Delta_2' v = v_m\ f(\sqrt{\lambda}) = v_0 \frac{v_m}{v_0}\ f(\sqrt{\lambda}) = v_0\ f(\sqrt{\lambda})/\lambda^{1/4}$$

Combining,

$$\frac{\Delta v + \Delta_2 v + \Delta' v + \Delta_2' v}{v_0} = f(\sqrt{\lambda})(1 + 1/\lambda^{1/4})$$

The ratio C of the 4-boost to the two-boost rocket energy (proportional to the sum of the Δv's) is evidently the right hand side divided by $f(\lambda)$:

$$C = \frac{f(\sqrt{\lambda})}{f(\lambda)}(1 + 1/\lambda^{1/4})$$

The curves below are numerical plots of $C(\lambda)$ vs λ (upper curve) and $f(\sqrt{\lambda})/f(\lambda)$ (lower curve). The upper curve is always greater than 1, showing that the 4-boost method is always more expensive.