Physics 185 Problem Set 7 — Due Wednesday 22 Feb. 2012 in class

new wording is indicated by •.

7.1 (10 points) Larmor precession

A classical particle of charge q is orbiting around a •fixed opposite charge in the presence of a weak magnetic field B perpendicular to the orbit.

a) Find a •uniformly-rotating co-ordinate system in which the Lorentz force $q\dot{r} \times B$ vanishes. •That is, determine the magnitude and direction of the angular velocity Ω of the rotating frame.

b) •In the rotating frame there is still a residual effect of the magnetic field: a residual force. Find the residual force on the particle in this frame. It is quadratic in B.

c) Supposing that B is so small that the residual force in b) can be neglected, what is the effect of B as viewed from the rest frame?

7.2 (12 points) Translation of axis of rotation

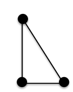
A rigid body can be viewed as a set of point masses m_α at positions $\vec{r}_\alpha$. A body has angular momentum $\vec{L}_0$ when rotating at angular velocity $\vec{\omega}$ about its center of mass. If the axis rotation is shifted from the center of mass by displacement $\vec{R}$, but $\vec{\omega}$ remains fixed, $\vec{L}$ changes from its initial value $\vec{L}_0$. For this problem it is useful to know the "BAC - CAB" formula: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$.

a) What is the change of $\vec{L}$ (*i.e.*, $\vec{L} - \vec{L}_0$) that results from shifting the origin?

b) How does your answer in a) change if the entire mass M is concentrated at a point?

c) If $\vec{\omega}$ is along the z axis, what is the change of I_{zz}, I_{zx} and I_{zy} as a function of $\vec{R}_x$, $\vec{R}_y$ and $\vec{R}_z$?

d) What is the change in kinetic energy T when the origin is shifted? Is there an R for which the kinetic energy is minimal?



7.3 (12 points) Finding principal axes

A rigid body consists of three point masses m arranged in a 3-4-5 right triangle with hypotenuse $5a$. The triangle is oriented in the x-y plane so that its shortest side is along the $+x$ axis as shown. We wish to find the principal axes of the inertia tensor $\mathbf{I}$ for rotation around the top vertex. Evidently if $\vec{\omega}$ is along z, so is the $\vec{\ell}_\alpha$ contribution to $\vec{L}$ from each mass. Thus z is a principal axis. The other two principal axes lie in the x-y plane rotated by some angle θ from the x axis.

a) Find I_{xx}, I_{xy}, I_{yx} and I_{yy}.

b) Find the principal moments in the x-y plane, denote I_+ and I_-.

c) Find the angle θ for the principal axis with the larger moment relative to the short side of the triangle.

Solution:

7.1

a) In a frame rotating with angular velocity $\vec{\Omega}$, there is an inertial force $2m\vec{\dot{r}} \times \vec{\Omega} + m\Omega^2 r\hat{r}$. The orbital speed $\vec{\dot{r}}$ in the rotating frame is related to the inertial velocity $\vec{v}$ by $\vec{\dot{r}} = \vec{v} - \vec{\Omega} \times \vec{r}$. The magnetic force in the rotating frame is thus $q\vec{v} \times \vec{B} = (\vec{\dot{r}} + \Omega \times \vec{r}) \times \vec{B}$. Thus the effective force F_{eff} including the magnetic force and the inertial forces is

$$2m\vec{\dot{r}} \times \vec{\Omega} + m\Omega^2 r\hat{r} + q(\vec{\dot{r}} + \Omega \times \vec{r}) \times \vec{B}$$

Re-arranging

$$F_{\text{eff}} = \vec{\dot{r}} \times (2m\vec{\Omega} + q\vec{B}) + m\Omega^2 r\hat{r} + q(\vec{\Omega} \times \vec{r}) \times \vec{B}$$

By setting $\vec{\Omega} = -q/(2m)\vec{B}$ we eliminate the Lorentz-force term $\vec{\dot{r}} \times (...)$.

b) What remains is the centrifugal force proportional to $\Omega^2 \sim B^2$ and the $\vec{\Omega} \times \vec{r} \times \vec{B}$ force, also proportional to B^2. Specifically,

$$F_{\text{eff}} = \frac{q^2 B^2}{4m} r\ \hat{r} - \frac{q^2}{2m}(\vec{B} \times \vec{r}) \times \vec{B}$$

Using BAC - CAB formula

$$F_{\text{eff}} = \frac{q^2 B^2}{4m} r\ \hat{r} + \frac{q^2}{2m}[\vec{B}(\vec{B} \cdot \vec{r}) - \vec{r}(\vec{B} \cdot \vec{B})]$$

Since $\vec{B}$ was assumed to be perpendicular to the orbit, the $\vec{B} \cdot \vec{r} = 0$. Combining,

$$F_{\text{eff}} = -\frac{q^2 B^2}{4m} r\ \hat{r}$$

c) In the rotating frame, there is no effect from the B field (since we are ignoring the effects that are quadratic in B). Thus in the rest frame, the elliptical orbit gradually rotates or precesses about the B axis.

7.2

a)

$$\vec{L}_0 = \sum_\alpha m_\alpha\ \vec{r}_\alpha \times (\vec{\omega} \times r_\alpha)$$

$$\vec{L} = \sum_\alpha m_\alpha\ (\vec{r}_\alpha + \vec{R}) \times (\vec{\omega} \times (\vec{r}_\alpha + \vec{R}))$$

$$\vec{L} = \sum_\alpha m_\alpha\ \left[\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) + \vec{r}_\alpha \times (\vec{\omega} \times \vec{R}) + \vec{R} \times (\vec{\omega} \times \vec{r}_\alpha) + \vec{R} \times (\vec{\omega} \times \vec{R})\right]$$

Expanding,

$$\vec{L} = \vec{L}_0 + (\sum_\alpha m_\alpha\ \vec{r}_\alpha) \times (\vec{\omega} \times \vec{R}) + \vec{R} \times (\vec{\omega} \times (\sum_\alpha m_\alpha\ \vec{r}_\alpha)\ ) + (\sum_\alpha m_\alpha)\ \vec{R} \times (\vec{\omega} \times \vec{R})$$

But since $\vec{r}_\alpha$ was measured from the center of mass, $\sum_\alpha m_\alpha\ \vec{r}_\alpha = \vec{0}$, so that

$$\vec{L} = \vec{L}_0 + M\ \vec{R} \times (\vec{\omega} \times \vec{R})$$

You may expand this using the BAC-CAB formula, though it isn't necessary here. This gives

$$\vec{L} - \vec{L}_0 = M \left(\vec{\omega}\ R^2 - \vec{R}\ (\vec{\omega} \cdot \vec{R}) \right)$$

b) From a) we see that $\vec{L} - \vec{L}_0$ is independent of the distribution of mass. Thus the answer is *the same* if all the mass is concentrated in the center.

c)

$$\vec{L} = \omega \left(I_{zz}\ \hat{z} + I_{xz}\ \hat{x} + I_{yz}\ \hat{y} \right)$$

likewise

$$\vec{L}_0 = \omega \left(I_{0zz}\ \hat{z} + I_{0xz}\ \hat{x} + I_{0yz}\ \hat{y} \right)$$

Comparing with b)

$$I_{zz} - I_{0zz} = M\ \hat{z} \cdot [\vec{R} \times (\hat{z} \times \vec{R})] = M\ \hat{z} \cdot [\hat{z}\ R^2 - \vec{R}\ (\hat{z} \cdot \vec{R})] = M\ [R^2 - (\hat{z} \cdot \vec{R})^2]$$

$$I_{xz} - I_{0xz} = M\ \hat{x} \cdot [\hat{z}\ R^2 - \vec{R}\ (\hat{z} \cdot \vec{R})] = -M\ (\hat{x} \cdot \vec{R})(\hat{z} \cdot \vec{R})$$

$$I_{yz} - I_{0yz} = M\ \hat{y} \cdot [\hat{z}\ R^2 - \vec{R}\ (\hat{z} \cdot \vec{R})] = -M\ (\hat{y} \cdot \vec{R})(\hat{z} \cdot \vec{R})$$

d)

$$\Delta T = \frac{1}{2}\vec{\omega} \cdot (\vec{L} - \vec{L}_0) = \frac{1}{2}M\ \vec{\omega} \cdot [\vec{\omega}\ R^2 - \vec{R}\ (\vec{\omega} \cdot \vec{R})] = \frac{1}{2}M\ [(\omega R)^2 - (\vec{\omega} \cdot \vec{R})^2]$$

Since $|\omega R|$ is always greater than $|\omega \cdot R|$, ΔT is always greater than zero. ($\Delta T = 0$ whenever $\vec{R} \parallel \vec{\omega}$, since such a shift does not change the axis.) Thus the minimum T occurs when the origin is at the center of mass.

7.3

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a) Only the bottom two masses contribute to I. For rotation about the y axis, only the mass on x axis contributes

$$I_{yy} = m\ a^2 3^2 \quad ; \quad I_{xy} = I_{yx} = -ma^2(3 * 4)$$

For rotation about the x axis

$$I_{xx} = 2m\ a^2 4^2$$

Thus the I matrix in the x-y plane is given by

$$I = m\ a^2 \begin{bmatrix} 32 & -12 \\ -12 & 9 \end{bmatrix}$$

b) Denoting the numerical matrix as $\hat{I}$ and its eigenvalues $\hat{I}_\pm$,

$$\det(\hat{I} - \hat{I}_\pm\ \mathbf{1}) = 0$$

$$\det \begin{bmatrix} 32 - \hat{I}_\pm & -12 \\ -12 & 9 - \hat{I}_\pm \end{bmatrix} = 0$$

expanding the det we obtain

$$0 = (32 - \hat{I}_\pm)(9 - \hat{I}_\pm) - 144$$

or

$$0 = (32 * 9 - 144) - (32 + 9)I_\pm + I_\pm^2$$

whose roots are

$$\hat{I}_{\pm} = \frac{(32+9) \pm \sqrt{(32+9)^2 - 4(32*9-144)}}{2} = \frac{41 \pm \sqrt{1681-576}}{2}$$

Numerically

$$\hat{I}_{\pm} = 37.1..., 3.88... \quad ; \quad I_{\pm} = ma^2(37.1..., 3.88...;)$$

The principal values always bracket the diagonal values.

c) Now that we know the principal eigenvalues, we can solve for the eigenvectors, *eg.* using $\hat{I}_+$,

$$(32 - \hat{I}_+)\omega_x - 12\omega_y = 0$$

so that

$$\tan\theta = \omega_y/\omega_x \simeq (32-37.1)/12 \simeq -0.403$$

so that $\theta \simeq -23.1$ degrees.