8.1 (7 points) spinning slab

A thin slab or "lamina" has all its mass in the x-y plane, like a sheet of cardboard cut into some random shape. We consider the inertia tensor $\mathbf{I}$ expressed in the x-y-z co-ordinates about an arbitrary origin point in the x-y plane.

a) Is the z axis a principal axis for $\mathbf{I}$? Why or why not?

b) There is a simple relation between $\mathbf{I}_{zz}$, $\mathbf{I}_{xx}$ and $\mathbf{I}_{yy}$ for any lamina. What is it?

8.2 (13 points) discarded CD

You throw an unwanted CD out the window. (The CD may be considered a lamina as in the previous problem.) You are not throwing at your best and you launch the CD with an angular velocity ω at a slight angle α with respect to the symmetry axis normal to the CD plane. Thus a small mark on the rim makes a full rotation in a time $2\pi/\omega$. So the CD wobbles as it flies: the CD axis rotates about the angular momentum direction in a time τ_w.

a) Using the previous problem, what is the ratio between the $\mathbf{I}_{11}$ along the CD axis and $\mathbf{I}_{33}$ around an axis in the disk.

b) What is the angle β between the axis and the angular momentum $\vec{L}$ in terms of α?

c) Express the kinetic energy $T = \frac{1}{2}\vec{\omega} \cdot \vec{L}$ in terms of $|L|$ and the angle β. How does β vary with time?

d) How do α and $|\omega|$ vary with time?

e) What is the relation between the wobble time τ_w and the spinning period $2\pi/\omega$? It's ok to use the relationship proved in Problem 10.46: $\tau_w = 2\pi\lambda_1/|L|$, where λ_1 is the principal moment perpendicular to the CD axis.

Solution:

8.1

a) To see if z is a principal axis, we compute $\mathbf{I}\ \hat{z}$ and see if it is a multiple of $\hat{z}$. If so, then $\hat{z}$ is an eigenvector, and z is a principal axis. Now by the rules of matrix multiplication

$$\mathbf{I}\ \hat{z} = \mathbf{I}_{xz}\hat{x} + \mathbf{I}_{yz}\hat{y} + \mathbf{I}_{zz}\hat{z}$$

$\mathbf{I}_{xz}$ is a product of inertia $= \sum_\alpha m_\alpha x_\alpha z_\alpha$. But in this object all the mass has $z_\alpha = 0$, so $\mathbf{I}_{xz} = 0$ similarly, $\mathbf{I}_{yz} = 0$. Thus $\mathbf{I}\ \hat{z} = \mathbf{I}_{zz}\hat{z}$ It is proportional to $\hat{z}$. thus z is a principal axis.

b) We write out the definition $\mathbf{I}_{zz}$, $\mathbf{I}_{xx}$ and $\mathbf{I}_{yy}$ using the fact that $z_\alpha = 0$ for all the masses making up the object:

$$\mathbf{I}_{zz} = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2) \quad \mathbf{I}_{xx} = \sum_\alpha m_\alpha(y_\alpha^2 + 0) \quad \mathbf{I}_{yy} = \sum_\alpha m_\alpha(x_\alpha^2 + 0)$$

Evidently $\mathbf{I}_{zz} = \mathbf{I}_{xx} + \mathbf{I}_{yy}$

8.2

a) We take the cd axis to be the $\hat{e}_3$ direction. Then $\hat{e}_2$ and $\hat{e}_1$ are in the cd plane. Using 8.1b and the equivalence of all directions in the cd plane,

$$\mathbf{I}_{33} = \mathbf{I}_{11} + \mathbf{I}_{22} = 2\mathbf{I}_{11}$$

b) The 1, 2, and 3 axes are principal axes, so $L = \mathbf{I}_{11}\omega_1\ \hat{e}_1 + \mathbf{I}_{22}\omega_2\ \hat{e}_2 + \mathbf{I}_{33}\omega_3\ \hat{e}_3$. Let us choose the 1 and 2 axes so that $\omega_2 = 0$. Then $L_2 = 0$ as well, and $L = \mathbf{I}_{33}(\mathbf{I}_{11}/\mathbf{I}_{33}\ \omega_1\ \hat{e}_1 + \omega_3\ \hat{e}_3$. We may find the angle β by

$$\tan\beta = \frac{L_1}{L_3} = \left(\frac{\mathbf{I}_{11}}{\mathbf{I}_{33}}\right)\ \frac{\omega_1}{\omega_3} = \left(\frac{\mathbf{I}_{11}}{\mathbf{I}_{33}}\right)\tan\alpha = \frac{1}{2}\tan\alpha$$

For small α, $\beta = \frac{1}{2}\alpha$.

c) $L_i = \mathbf{I}_{ii}\omega_i$, so $\omega_i = 1/\mathbf{I}_{ii}\ L_i$. Thus

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}{L_1}^2/\mathbf{I}_{11}{L_2}^2/\mathbf{I}_{22} + {L_3}^2/\mathbf{I}_{33}.$$

In our chosen co-ordinates $L_2 = 0$, so

$$T = \frac{1}{2}\frac{{L_3}^2}{\mathbf{I}_{33}}\Big(1 + (L_1/L_3)^2\mathbf{I}_{33}/\mathbf{I}_{11})\Big) = \frac{(|L|/\cos\beta)^2}{\mathbf{I}_{33}}(1 + 2\tan^2\beta)$$

Since T, L_3 and $\mathbf{I}_{33}$ are constant in time, β must be also.

d) The argument of b) that $\frac{1}{2}\tan\alpha = \tan\beta$ is true for all times. Since β is constant in time, so is α. $\omega_3 = L_3/\mathbf{I}_{33}$. The right side is constant in time, so ω_3 is constant. Thus $|\omega| = \omega_3/\tan\alpha$ is also constant.

e)

$$\tau_w = 2\pi\frac{\lambda_1}{|L|} = 2\pi\frac{\mathbf{I}_{11}}{|L|} = (2\pi)\ \frac{\frac{1}{2}\mathbf{I}_{33}}{|L|} = (2\pi)\ \frac{1}{2}\cos\beta(\mathbf{I}_{33}/L_3) = (2\pi)\ \frac{1}{2}\cos\beta/\omega_3 = (2\pi)\ \frac{1}{2}\frac{\cos\beta}{\omega\cos\alpha}$$

Since α is assumed small, $\cos\alpha \simeq 1$, $\beta \propto \alpha$ is also small, so $\cos\beta \simeq 1$ as well.

$$\tau_w \simeq \frac{1}{2}(2\pi/\omega)$$

The axis wobbles twice in every rotational time.