

Three misprints in the original solution are corrected and marked by •.

8.1 (7 points) spinning slab

A thin slab or “lamina” has all its mass in the x-y plane, like a sheet of cardboard cut into some random shape. We consider the inertia tensor \mathbf{I} expressed in the x-y-z co-ordinates about an arbitrary origin point in the x-y plane.

- a) Is the z axis a principal axis for \mathbf{I} ? Why or why not?
- b) There is a simple relation between \mathbf{I}_{zz} , \mathbf{I}_{xx} and \mathbf{I}_{yy} for any lamina. What is it?

8.2 (13 points) discarded CD

You throw an unwanted CD out the window. (The CD may be considered a lamina as in the previous problem.) You are not throwing at your best and you launch the CD with an angular velocity ω at a slight angle α with respect to the symmetry axis normal to the CD plane. Thus a small mark on the rim makes a full rotation in a time $2\pi/\omega$. So the CD wobbles as it flies: the CD axis rotates about the angular momentum direction in a time τ_w .

- a) Using the previous problem, what is the ratio between the \mathbf{I}_{11} along the CD axis and \mathbf{I}_{33} around an axis in the disk.
- b) What is the angle β between the axis and the angular momentum \vec{L} in terms of α ?
- c) Express the kinetic energy $T = \frac{1}{2}\vec{\omega} \cdot \vec{L}$ in terms of $|L|$ and the angle β . How does β vary with time?
- d) How do α and $|\omega|$ vary with time?
- e) What is the relation between the wobble time τ_w and the spinning period $2\pi/\omega$? It’s ok to use the relationship proved in Problem 10.46: $\tau_w = 2\pi\lambda_1/|L|$, where λ_1 is the principal moment perpendicular to the CD axis.

Solution:

8.1

- a) To see if z is a principal axis, we compute $\mathbf{I} \hat{z}$ and see if it is a multiple of \hat{z} . If so, then \hat{z} is an eigenvector, and z is a principal axis. Now by the rules of matrix multiplication

$$\mathbf{I} \hat{z} = \mathbf{I}_{xz}\hat{x} + \mathbf{I}_{yz}\hat{y} + \mathbf{I}_{zz}\hat{z}$$

\mathbf{I}_{xz} is a product of inertia = $\sum_{\alpha} m_{\alpha}x_{\alpha}z_{\alpha}$. But in this object all the mass has $z_{\alpha} = 0$, so $\mathbf{I}_{xz} = 0$ similarly, $\mathbf{I}_{yz} = 0$. Thus $\mathbf{I} \hat{z} = \mathbf{I}_{zz}\hat{z}$ It is proportional to \hat{z} . thus z is a principal axis.

- b) We write out the definition \mathbf{I}_{zz} , \mathbf{I}_{xx} and \mathbf{I}_{yy} using the fact that $z_{\alpha} = 0$ for all the masses making up the object:

$$\mathbf{I}_{zz} = \sum_{\alpha} m_{\alpha}(x_{\alpha}^2 + y_{\alpha}^2) \quad \mathbf{I}_{xx} = \sum_{\alpha} m_{\alpha}(y_{\alpha}^2 + 0) \quad \mathbf{I}_{yy} = \sum_{\alpha} m_{\alpha}(x_{\alpha}^2 + 0)$$

Evidently $\mathbf{I}_{zz} = \mathbf{I}_{xx} + \mathbf{I}_{yy}$

8.2

- a) We take the cd axis to be the \hat{e}_3 direction. Then \hat{e}_2 and \hat{e}_1 are in the cd plane. Using 8.1b and the equivalence of all directions in the cd plane,

$$\mathbf{I}_{33} = \mathbf{I}_{11} + \mathbf{I}_{22} = 2\mathbf{I}_{11}$$

- b) The 1, 2, and 3 axes are principal axes, so $L = \mathbf{I}_{11}\omega_1 \hat{e}_1 + \mathbf{I}_{22}\omega_2 \hat{e}_2 + \mathbf{I}_{33}\omega_3 \hat{e}_3$. Let us choose the 1 and 2 axes so that $\omega_2 = 0$. Then $L_2 = 0$ as well, and $L = \mathbf{I}_{33}(\mathbf{I}_{11}/\mathbf{I}_{33} \omega_1 \hat{e}_1 + \omega_3 \hat{e}_3)$. We may find the angle β by

$$\tan \beta = \frac{L_1}{L_3} = \left(\frac{\mathbf{I}_{11}}{\mathbf{I}_{33}} \right) \frac{\omega_1}{\omega_3} = \left(\frac{\mathbf{I}_{11}}{\mathbf{I}_{33}} \right) \tan \alpha = \frac{1}{2} \tan \alpha$$

For small α , $\beta = \frac{1}{2}\alpha$.

- c) $L_i = \mathbf{I}_{ii}\omega_i$, so $\omega_i = 1/\mathbf{I}_{ii} L_i$. Thus

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} L_1^2 / \mathbf{I}_{11} + L_2^2 / \mathbf{I}_{22} + L_3^2 / \mathbf{I}_{33}.$$

In our chosen co-ordinates $L_2 = 0$, so

$$T = \frac{1}{2} \frac{L_3^2}{\mathbf{I}_{33}} \left(1 + (L_1/L_3)^2 \mathbf{I}_{33}/\mathbf{I}_{11} \right) = \frac{(|L| \bullet \cos \beta)^2}{\mathbf{I}_{33}} (1 + 2 \tan^2 \beta)$$

Since T , L_3 and \mathbf{I}_{33} are constant in time, β must be also.

- d) The argument of b) that $\frac{1}{2} \tan \alpha = \tan \beta$ is true for all times. Since β is constant in time, so is α . $\omega_3 = L_3/\mathbf{I}_{33}$. The right side is constant in time, so ω_3 is constant. Thus $|\omega| = \omega_3 / \bullet \cos \alpha$ is also constant.

- e)

$$\tau_w = 2\pi \frac{\lambda_1}{|L|} = 2\pi \frac{\mathbf{I}_{11}}{|L|} = (2\pi) \frac{\frac{1}{2}\mathbf{I}_{33}}{|L|} = (2\pi) \frac{1}{2} \cos \beta (\mathbf{I}_{33}/L_3) = (2\pi) \frac{1}{2} \cos \beta / \omega_3 = (2\pi) \frac{1}{2} \frac{\cos \beta}{\omega \cos \alpha}$$

Since α is assumed small, $\cos \alpha \simeq 1$, $\beta \propto \alpha$ is also small, so $\cos \beta \simeq 1$ as well.

$$\tau_w \simeq \frac{1}{2} (2\pi/\omega)$$

The axis wobbles twice in every rotational time.