9.1 (4 points) conjugate momentum and dilation

Suppose I dilate my co-ordinate q_1 by a factor λ. while leaving the others untouched? Thus my new coordinates are

$$(Q_1, Q_2, ...Q_N) = (\lambda q_1, q_2, ...q_N)$$

What is the relation between the new conjugate momenta $P_1, ...P_n$ and the original $p_1, ..., p_N$?

9.2 (13 points) Poisson brackets

In class we defined a mathematical operation called the "Poisson Bracket". Given any two functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$, the Poisson bracket $[f, g]$ is another function of $\mathbf{q}, \mathbf{p}$ defined as

$$[f, g] = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

We noted that for a given f, the poisson bracket with the Hamiltonian $[f, \mathcal{H}] = \dot{f}$. *i.e.,* for an infinitesimal time interval Δt,

$$f(t + \Delta t) = f(t) + [f, \mathcal{H}]\Delta t$$

We say that $\mathcal{H}$ *generates* an infinitesimal time translation. We saw earlier in the course the relation between $\mathcal{H}$ and time translation.

a) apply the poisson bracket to the case where f is simply one of the q's or p's, say q_1 or p_1. Does this agree with your prior information about $\dot{q}_1$ and $\dot{p}_1$?

b) Clearly $[g, f]$ gives the same information as $[f, g]$. What is the mathematical relation between these two quantities? What can you say about $[f, f]$?

c) What is the value of $[q_i, p_i]$?

d) Consider a single particle of mass m moving in the $x - y$ plane, where the potential energy $U(x, y)$ depends only on position. Consider the polar co-ordinates, r, and ϕ and the corresponding cartesian co-ordinates $x = r\cos\phi$, and $y = r\sin\phi$. We saw earlier that p_ϕ is the momentum that is conserved when the system is invariant under rotation just as the Hamiltonian is conserved when the system (not the motion) is independent of time. In this connection it is interesting to see what sort of transformation p_ϕ generates. Find p_x and p_y in terms of r, ϕ, p_r and p_ϕ. and then

e) find $[x, p_\phi]$ and $[p_x, p_\phi]$. Compare with law for the infinitesimal rotation of any vector $\vec{v}$

$$v_x(\phi + \Delta\phi) = v_x(\phi) - v_y\Delta\phi \quad ; \quad v_y(\phi + \Delta\phi) = v_y(\phi) + v_x\Delta\phi$$

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9.3 (8 points) Canonical Co-ordinates

Here I use the the book's notation $\mathbf{q}$, $\mathbf{p}$ rather than my lecture notation $\{q\}$, $\{p\}$.

The Poisson Bracket transformation presumes a complete set of $\mathbf{q}$'s and their conjugate $\mathbf{p}$'s. We showed in class that $[f, \mathcal{H}] = \dot{f}$ without specifying what the co-ordinates $\mathbf{q}$ were (and independent of the choice of the function $\mathcal{H}$). That means we would have got the same $\dot{f}$ regardless of what $\mathbf{q}$'s we had started with. We are free to switch to $\mathbf{Q}$'s that are a function of the $\mathbf{q}$, and the corresponding poisson brackets $[\![f, g]\!]$ using these $\mathbf{Q}$ and *their* conjugate momenta $\mathbf{P}$ gives the same function as the original $[f, g]$. That is,

$$[\![f, g]\!] \equiv \sum_{i=1}^{N} \frac{\partial f}{\partial Q_i}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial Q_i} \quad = \quad [f, g] \equiv \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Any set of co-ordinates $\mathbf{Q}$ and $\mathbf{P}$ with this property are a "canonical" set of co-ordinates for phase space. They are equivalent to the original set $\mathbf{q}, \mathbf{p}$ for computing whatever quantity we like.

A big advantage of phase space over co-ordinate space is that it gives added flexibility in choosing co-ordinates. *Any* $\mathbf{Q}, \mathbf{P}$ that gives $[\![f, g]\!] = [f, g]$ is fine. In particular we could choose $\mathbf{Q}$ to be a function of *both* $\mathbf{q}$ and $\mathbf{p}$ (rather than just $\mathbf{q}$ as above). To establish this property, we don't need to examine all f and g. It is sufficient to show it for a complete set of co-ordinates, *i.e.*, for f of g equal to any of the q_i or p_i. That is, we only need to show

$$[\![q_i, q_j]\!] = [q_i, q_j] = 0 \;\; ; \;\; [\![p_i, p_j]\!] = [p_i, p_j] = 0 \;\; ; \;\; [\![q_i, p_j]\!] = [q_i, p_j] = \delta_{ij} \tag{1}$$

Notice that once we have found one set of canonical co-ordinates, (*viz.* $\mathbf{q}$ and $\mathbf{p}$) we can find others without further reference to the Lagrangian or Hamiltonian. The new co-ordinates only have to preserve the quantity $[f, g]$ that was specified using *eg.* the Cartesian co-ordinates of our system and Newton's equation.

Consider a system with mass m and (Cartesian) position x, so that $p = m\dot{x}$. We define new variables Q and P by

$$x = \sqrt{2P}\sin Q \quad ; \quad p = \sqrt{2P}\cos Q$$

a) Determine whether Q, P are canonical according to (1). Since $[\![x, x]\!] = [\![p, p]\!] = 0$ automatically, you only need to check whether $[\![x, p]\!] = 1$.

b) Now suppose that the system is a harmonic oscillator with mass 1, so that $\mathcal{H} = \frac{1}{2}(x^2 + p^2)$. Express $\mathcal{H}$ in terms of Q and P. Does one of the variables become ignorable? Which? What is the corresponding conserved quantity?

Sometimes one can find a set of $\mathbf{Q}, \mathbf{P}$ analogous to Q and P above. The P_i, Q_i are called *action-angle* variables. Any system (with a given $\mathcal{H}$) with a complete set of action-angle variables clearly has a complete set of ignorable co-ordinates and a complete set of conserved momenta. Such systems are called *integrable*, since their motion is completely solved in terms of the action-angle variables.. For example, central-force motion as described in Chapter 8 is integrable, but motion with a general two-dimensional potential is not.

Solution:

8.1

$P_1 = \partial\mathcal{L}/\partial Q_1 = 1/\lambda \partial\mathcal{L}/\partial q_1 = p_1/\lambda$. The other P's are unchanged: $P_i = p_i$ for $i > 1$.

8.2

a)

$$[f, g] = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

$$[q_1, \mathcal{H}] = \sum_{i=1}^{N} \frac{\partial q_1}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial q_1}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}$$

The first term vanishes unless $i = 1$. (because if $i \neq 1$, $\partial\ /\partial q_i$ means one is supposed the other q's and p's constant, including q_1. The second term vanishes altogether for the same reason. We are left with

$$[q_1, \mathcal{H}] = \frac{\partial \mathcal{H}}{\partial p_i}$$

That's just Hamilton's equation for $\dot{q}_1$. So the formula that $[q_1, \mathcal{H}]$ reduces to the familiar Hamilton's equation for $\dot{q}_1$.

b) $[g, f] = -[f, g]$; $[f, f] = -[f, f] = 0$.

c) Using the $[f, g]$ formula above and the meaning of ∂ noted above, it is clear that $[q_i, p_i] = 1$.

d) We know from earlier chapters that in polar co-ordinates the kinetic energy $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2$. The potential energy is independent of $\dot{\phi}$. Thus $p_r = m\dot{r}$, and $p_\phi = \partial T/\partial\dot{\phi} = mr^2\dot{\phi}$. Now we can express p_x, p_y in terms of this p_r and p_ϕ and the co-ordinates r and ϕ:

$$p_x = m\dot{x} = m(d/dt)r\cos\phi = m\dot{r}\cos\phi - mr\sin\phi\dot{\phi}$$

$$= m(p_r/m)\cos\phi - mr\sin\phi(p_\phi/mr^2) = \cos\phi\ p_r - (1/r)\sin\phi\ p_\phi$$

$$p_y = m\dot{y} = m(d/dt)r\sin\phi = m\dot{r}\sin\phi + mr\cos\phi\dot{\phi}$$

$$= m(p_r/m)\sin\phi + mr\cos\phi(p_\phi/mr^2) = \sin\phi\ p_r + (1/r)\cos\phi\ p_\phi$$

e)

$$[x, p_\phi] = \frac{\partial x}{\partial r}\frac{\partial p_\phi}{\partial p_r} - \frac{\partial x}{\partial p_r}\frac{\partial p_\phi}{\partial r} + \frac{\partial x}{\partial \phi}\frac{\partial p_\phi}{\partial p_\phi} - \frac{\partial x}{\partial p_\phi}\frac{\partial p_\phi}{\partial \phi}$$

The only nonzero ∂p_ϕ is $\partial p_\phi/\partial p_\phi = 1$ Thus

$$[x, p_\phi] = \partial x/\partial\phi = -r\sin\phi = -y.$$

Likewise

$$[y, p_\phi] = \partial y/\partial\phi = +r\cos\phi = x$$

$$[p_x, p_\phi] = \frac{\partial p_x}{\partial r}\frac{\partial p_\phi}{\partial p_r} - \frac{\partial p_x}{\partial p_r}\frac{\partial p_\phi}{\partial r} + \frac{\partial p_x}{\partial \phi}\frac{\partial p_\phi}{\partial p_\phi} - \frac{\partial p_x}{\partial p_\phi}\frac{\partial p_\phi}{\partial \phi}$$

$$\frac{\partial p_x}{\partial \phi} = -\sin\phi\ p_r - \cos\phi/r\ p_\phi = -p_y$$

So $[\ , p_\phi]$ transforms both x and p_x by an infinitesimal rotation.

8.3

a)

$$\frac{\partial x}{\partial Q} = \sqrt{2P}\cos Q \quad ; \quad \frac{\partial x}{\partial P} = \sqrt{2}\ \frac{1}{2}\ \frac{1}{\sqrt{P}}\sin Q = \ \frac{1}{\sqrt{2P}}\sin Q \quad ;$$

$$\frac{\partial p}{\partial Q} = -\sqrt{2P}\sin Q \quad ; \quad \frac{\partial p}{\partial P} = \sqrt{2}\ \frac{1}{2}\ \frac{1}{\sqrt{P}}\cos Q = \ \frac{1}{\sqrt{2P}}\cos Q$$

Thus

$$[\![x, p]\!] \equiv \frac{\partial x}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial x}{\partial P}\frac{\partial p}{\partial Q} = (\sqrt{2P}\cos Q)\frac{1}{\sqrt{2P}}\cos Q - \frac{1}{\sqrt{2P}}\sin Q(-\sqrt{2P}\sin Q)$$

Simplifying,

$$[\![x, p]\!] = \cos^2 Q + \sin^2 Q = 1$$

So Q, P are canonical. (I didn't show in class that this condition guarantees that the P and Q are canonical.)

b)

$$x^2 = 2P\sin^2 Q \quad ; \quad p^2 = 2P\cos^2 Q$$

so that

$$\mathcal{H} = \frac{1}{2}(x^2 + p^2) = \frac{1}{2}\ (2P)(\sin^2 Q + \cos^2 Q) = P$$

We see that the canonical momentum is $\mathcal{H}$ itself. Since Q is ignorable, this P is conserved. This is how the conservation of energy emerges in this language.